The Multi-round Process Matrix

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1 Introduction

Quantum communication protocols usually involve several parties acting successively on a quantum system as a means of exchanging information. However, it was found that more general situations in which the order of the operations of the parties is subject to quantum indefiniteness are also conceivable [10, 7]. This is the case, for example, when two parties exchange a system in a direction that is coherently controlled by another quantum system, leading to information being conveyed in a superposition of both directions. Such a situation is an example of a *causally nonseparable* process [11, 12], *i.e.* one that cannot be understood as a probabilistic mixture of different sequencing of the parties' operations.

There has been growing interest in this phenomenon lately. On the foundational side, causal nonseparability challenges established notions of causality and opens the possibility for investigating new theoretical possibilities [10, 7, 11, 4]. On the applied side, it promises new applications in information processing, having been shown to enable tasks that cannot be achieved with definite causal structures [5, 2, 8, 9, 13].

The representation of such processes can be done through the *Process Matrix* (PM) formalism [11], which extends the usual channel formalism by dropping the assumption of fixed causal ordering. This formalism however assume that each party in the process only have one round of information exchange with other parties, making it ill-equipped to describe communication protocols and other real-life applications. To fix this issue, our paper introduce a generalisation of PM called the *Multi-round Process Matrix* (MPM).

2 Results

As the name implies, the MPM relies on allowing the parties to have more than one round of communication (or action) during the process. The extension is obtained by grouping several parties together as one single local party and allowing her to have side communication outside of the process between each of her operations. Hence it is equivalent to a party communicating with the rest of the process several times while keeping a memory of her actions amid each round.

This extension is physically reasonable as nothing forbids a quantum system to be sent twice to the same party during the process. Moreover, the MPM is more apt to handle applications in practical communication protocols and distributed computing algorithms than the PM as multiple exchanges of information between parties as well as local communications are allowed.

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Mathematically, the extension is obtained by normalising the MPM on generalized quantum instruments, represented by the formalism of quantum combs [6], as opposed to normalising the PM on quantum instruments, which are represented by resolutions of quantum channels. Following the same argument as for the PM validity conditions [11], we find that an operator W is an MPM if it satisfies

$$W \geq 0 \quad , \tag{1}$$

$$\mathrm{Tr}\left[W \cdot \left(M^A \otimes M^B \otimes \ldots\right)\right] = 1 \quad , \tag{2}$$

where M^X is a deterministic quantum comb associated with party X's operations. Defined as such, the MPM admits the quantum comb as a special case when there is only a one party in the process, and it also admits the PM as a special case when each party is limited to one round of communication.

We then provide a full characterisation of the set of valid MPMs. To do so, we generalise a projective characterisation of the set of PM obtained in Ref. [1]. We proceed in 3 main steps: first we derive a projective validity conditions for deterministic quantum combs, then we do a similar characterisation of the set $\{M\}$ of all operators which are affine sums of tensors product of deterministic combs, *i.e.*

$$M \equiv \sum_i q_i \left(M^A \otimes M^B \ldots\right)_i \qquad , \sum_i q_i = 1 \qquad , \tag{3}$$

where the M^X are again deterministic combs associated with party X. Noticing that Eq. (2) implies that an MPM is also normalised on all M since the trace is linear, the final step amounts to deriving the projective characterisation of the MPM from the one of affine sums of combs. This last step is the content of Theorem 1 in our paper.

$$\begin{array}{l} M^X \geq 0 \\ \mathrm{Tr}\left[M^X\right] = d_{X_{in}} \\ \mathscr{P}^X\left\{M^X\right\} = M^X \end{array} \quad \underset{(3)}{\overset{\bigotimes_{X=A,B,\ldots}\ \text{composition}}{\Longrightarrow}} \quad \begin{array}{l} M \geq 0 \\ \mathrm{Tr}\left[M\right] = \prod_X d_{X_{in}} \\ \left(\bigotimes_X \mathscr{P}^X\right)[M] = M \end{array} \quad \underset{(1),(2)}{\overset{\text{Theorem 1}}{\Longrightarrow}} \quad \begin{array}{l} W \geq 0 \\ \mathrm{Tr}\left[W\right] = \prod_X d_{X_{out}} \\ \left(\mathscr{I} - \bigotimes_X \mathscr{P}^X + \mathscr{D}\right)[W] = W \end{array}$$

Table 1: The projective characterisation of a single comb (left) leads to the projective characterisation of $\{M\}$ (centre) using the tensor product composition of several parties. Theorem 1 then links this characterisation to the one of MPM (right). $\mathscr{P}^X$ is a superoperator projector acting on the subspace associated with party X, $\mathscr{I}$ is the identity supermap, $\mathscr{D}$ is the projector to the span of the unit matrix, and $d_{X_{in}}$ (resp. $d_{X_{out}}$) refers to the product of the dimensions of all input (output) subsystems of party X.

To complete the characterisation, we investigate the connection between MPMs and quantum combs. We prove in Theorem 2 that any MPM is an affine sum of deterministic quantum combs. For an MPM with a total of n operations, it can be decomposed into combs with $n+1$ teeth so that their first input and last output are trivial (see Fig. 1). These quantum combs have to be compatible with the local causal ordering of each party. In the example depicted in the figure, this means that the decomposition could not include some comb in which the second operation of A is before (graphically: below) her first one.

A natural ensuing question is to tell which MPMs are describing causal correlations between the parties. To answer this question, we formulate a (theory independent) notion of causality for the correlations within the process represented by an MPM, building upon the notion introduced in the original process framework [11]. This new definition allows us to speak about causal inequalities for the MPM and also guides us to define the (theory dependent) notion of causal separability for MPM. Motivated by the possibility to split a quantum comb into local single-round operations linked by side channels, we propose to define as causally separable the MPMs that, when extended by side channels between the consecutive operations of a same party, remain causally separable in the PM sense.

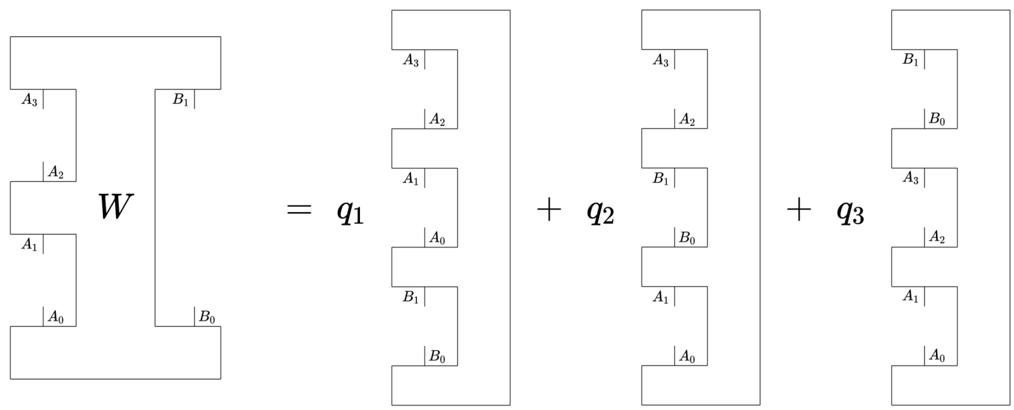

Figure 1: Graphical representation of Theorem 2 for an MPM shared by a party A having 2 operations and a party B having one. The MPM splits into 3 4-combs with different causal orderings, each respecting the constraint that the second operation of A cannot be before her first. Here, $\sum_i q_i = 1$.

This new definition bears non-trivial consequences. Since a PM is a special case of an MPM with no *a priori* partial ordering assumed at all, one may think that applying the PM definition to an MPM would be sufficient to establish causal (non)separability. Remarkably, this is not true, as we demonstrate by an example. We provide an operator that is both a valid PM and MPM but for which nonseparability can be *activated* in the latter case. That is, when the parties are considered to be able to communicate through the process matrix only, the situation is fully causally separable, no matter their actions. But as soon as the parties can use side-channels to share information, they can violate a causal inequality. This highlight a difference in the definitions as the formalism tells that a violation is possible, provided that we supply a wire in between two parties. However, this wire is not necessarily something that can be physically achieved in a lab, for instance in our example it would be oriented backwards in time.

3 Conclusion

We have developed an extension of the PM formalism that is more suitable to represent realistic communication protocols using indefinite causal order. Besides this practical aspect, the projective characterisation methods developed for the MPM are expected to give an easier way to characterise any object in the hierarchy of higher-order quantum processes [3].

Also, we have defined the notion of causal separability for the MPM and proved it to be different from the PM one. This difference, albeit being theoretic, sheds new light on the possibility of violating a causal inequality. It shows that considering the possibility that two parties could communicate, even if they do not, leads to a more restrictive notion. We anticipate this to be a guiding insight towards extending the concept of causal separability to higher-order processes, and understanding the physical realization of those.

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