Causality in definite and indefinite space-times

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1 Introduction

Given a fixed causal structure between events, Bell inequality violations can certify non-classicality of correlations produced by those events. This requires a precise understanding of the assumptions that rule out trivial Bell violations, and how they can be physically imposed. For example, if the parties in a CHSH Bell experiment could communicate during the experiment, or use pre-determined inputs, even classical strategies could trivially violate the CHSH inequality, and hence, the Bell inequality would not certify any non-classicality. A physical way to impose the non-signalling requirement is to place the parties in space-like separated locations, and a plausible way to avoid deterministic inputs is to have them be generated by 'good' local sources of randomness.

Going beyond the notion of a fixed background space-time or fixed causal structures, the possibility of indefinite causal orders has been widely studied [1–4]. Causal inequalities have been proposed for certifying the non-classicality of the causal structure itself; however, their physical and operational meaning is not yet well understood. Just like for Bell inequalities, a better physical intuition can be gained by characterising the assumptions behind causal inequalities.

For example, one such assumption is that the input to each local lab precedes its output ("*local order*") which is implicit in the process matrix framework [1], where causal inequalities are often studied. However, physically implementing this assumption — using measurements of local clocks to order these events — could decohere the process being considered when the events are delocalised in space-time [5,6]. To illustrate this, consider the protocol employed in the physical implementation [7, 8] of the *quantum switch* [2]. There, a target qubit is sent first to the lab of a party Alice and then to Bob's lab or vice-versa, depending coherently on the value of a control qubit, which is in a superposition. Alice's and Bob's local operations on the target inside their own labs are therefore performed at a superposition of different times, a process which is sometimes claimed to implement a superposition of causal orders. However, if either Alice or Bob measured their local time when they received the target state (to implement the local order assumption), they would decohere the superposition.

In the long run, we would like to have a good understanding of causality in indefinite space-time settings, where for example a superpositions of masses at different locations could lead to a superposition of different space-time geometries and hence, of causal orders [9]. A necessary first step is to study the possibility of indefinite causal and temporal orders within a fixed space-time. This will provide insights into the natural assumptions required for certifying non-classicality of causal orders through causal inequalities in both scenarios. This would also provide an avenue for exploring non-trivial physics within a fixed space-time, such as quantum fields that are not localised in space and time. Furthermore, experimental protocols within a fixed space-time background are more likely to be implemented in the near future. Finally, some such processes, like the natural generalisation of the quantum switch to N parties, are known to provide advantages over fixed order processes in certain information processing tasks [10], making them interesting both from foundational and practical perspectives.

Contribution. In this work, we compare the *causal box* (CB) [5] and *process matrix* (PM) [1] frameworks, where the former models events delocalised over a fixed space-time and the latter considers local quantum laboratories in the absence of a fixed background space-time. In particular, we analyse representations of the quantum switch in these two frameworks, and show that they are not equivalent: we identify the map from the CB to the PM representation and find that it is not invertible. This suggests physical distinctions between definite and indefinite space-time realisations [9] of the same process. Furthermore, we identify the assumptions required to rule out trivial causal inequality violations in the CB framework which allows for events to be delocalised within a fixed space-time, and show that bipartite CBs cannot violate causal inequalities under these assumptions, conjecturing the same for the multi-partite case.

2 Background

Here we provide a brief review of the causal box (CB) [5] and process matrix (PM) [1] frameworks, and the representations of the quantum switch in them, which will be required for stating our results.

2.1 Causal boxes [5]

The causal box formalism [5] models a general class of information-processing systems that respect causality and are closed under composition. The formalism assumes a fixed background space-time structure, and by including "space-time stamps" as part of every classical or quantum message, it can model situations where messages are sent at a superposition of different space-time locations. This allows it to model transformations such as the quantum switch (Section 2.3). Causal boxes can be seen as CPTP maps from space-time labelled quantum states on an input wire to those on an output wire, such that outputs at a given location can only depend on inputs in its causal past. We will not review all the details of the formal definition of causal boxes here, as this is very involved and not necessary for the purposes of this presentation. Instead, we will discuss some of the important concepts and properties involved in this definition to sketch an intuitive picture, referring the reader to [5] for the details.

Messages. Messages are modelled by qudits (where the message's quantum information is encoded), together with location information with respect to a partially ordered set $\mathcal{T}$ (for example, Minkowski space-time). The state space of a single d-dimensional *message* is $\mathcal{H} \coloneqq \mathbb{C}^d \otimes l^2(\mathcal{T})$ where $l^2(\mathcal{T})$ is the sequence space with a bounded 2-norm. An orthonormal basis for this space is $\{|v,t\rangle\}_{v\in\mathbb{C}^d, t\in\mathcal{T}}$, where $\{|v\rangle\}_{v\in\mathbb{C}^d}$ is an orthonormal basis of the qudit state-space $\mathbb{C}^d$.

Wires. Messages are carried by *wires* in and out of boxes. A d-dimensional wire can carry any number of d-dimensional messages or a superposition thereof. To model this, a wire's state space is the symmetric[1] Fock space $\mathcal{F}(\mathcal{H}) \coloneqq \bigoplus_{n=0}^{\infty} \vee^n\mathcal{H}$, where $\vee^n\mathcal{H}$ denotes the symmetric subspace of $\mathcal{H}^{\otimes n}$, and in particular $\mathcal{H}^{\otimes 0}$ is the one-dimensional space containing the vacuum state $|\Omega\rangle$. Some useful properties of this space are as follows [5]:

1. For two Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ of dimensions d_A and d_B respectively,
$$\mathcal{F}(\mathcal{H}_A \oplus \mathcal{H}_B) \cong \mathcal{F}(\mathcal{H}_A) \otimes \mathcal{F}(\mathcal{H}_B), \tag{1}$$
This allows for two (or more) wires to be combined into a single wire with the sum of the dimensions. Similarly, a single wire of dimensions d_A can be split into two (or more) wires whose dimensions sum to d_A.

2. If the state space corresponding to a wire A carrying d_A dimensional messages is denoted by $\mathcal{F}_A^{\mathcal{T}} = \mathcal{F}(\mathbb{C}^{d_A} \otimes l^2(\mathcal{T}))$, then for any subset $\mathcal{P} \subseteq \mathcal{T}$,
$$\mathcal{F}_A^{\mathcal{T}} \cong \mathcal{F}_A^{\mathcal{P}} \otimes \mathcal{F}_A^{\widetilde{\mathcal{P}}}, \tag{2}$$
where $\widetilde{\mathcal{P}} = \mathcal{T}\backslash\mathcal{P}$ and $\mathcal{F}_A^{\mathcal{P}} = \mathbb{C}^{d_A} \otimes l^2(\mathcal{P})$. In particular, this means that we can think of a state delocalised in space-time as living in a tensor product of Fock spaces, one associated with each disjoint region of space-time. We will use $\mathcal{F}_A^{\mathcal{P}}$ to denote both the space defined above, as well as its natural embedding in $\mathcal{F}_A^{\mathcal{T}}$ obtained by appending the vacuum state (representing "nothing") over $\widetilde{\mathcal{P}}$, i.e., $\mathcal{F}_A^{\mathcal{P}} \equiv \mathcal{F}_A^{\mathcal{P}} \otimes |\Omega\rangle_A^{\widetilde{\mathcal{P}}} \subseteq \mathcal{F}_A^{\mathcal{T}}$.

[1] The restriction to the symmetric subspace is to guarantee that there is no ordering information other than those given by the positions in $\mathcal{T}$ which is already included in the states [5].

Causality. A causal box is a system that performs a completely positive and trace preserving (CPTP) map from density matrices on an input wire X, $\mathfrak{T}(\mathcal{F}_X^{\mathcal{T}})$ to those on an output wire Y, $\mathfrak{T}(\mathcal{F}_Y^{\mathcal{T}})$, where $\mathfrak{T}(\mathcal{F})$ denotes the set of all trace class operators on the space $\mathcal{F}$.[2] A valid causal box must also satisfy a *causality condition*. Roughly speaking, this condition says that every output of a causal box only depends on inputs produced in its causal past. More formally, the description of a causal box producing outputs within a space-time region $\mathcal{C}$ (obtained by discarding outputs in $\mathcal{T}\backslash\mathcal{C}$) does not change if inputs produced outside the causal past of that region are discarded.

Composition of causal boxes. Causal boxes are closed under arbitrary composition [5]. Arbitrary composition operations can be built up from the following two types of composition: *parallel composition* (which produces a tensor product of causal boxes) and *loop composition* (linking outputs to future inputs by connecting the corresponding wires).

Representations of causal boxes. Being CPTP maps, causal boxes admit several different representations such as the Choi-Jamiołkowski and Stinespring dilation representations. They also allow for a new *sequence representation*, which depends on the space-time labels. It provides a decomposition of the causal box in terms of the operations in performs within subsequent, disjoint regions of space-time. We will see an example later on (Fig. 6).

2.2 Process matrices [1]

The process matrix framework [1] describes multi-partite scenarios where there is a local time order within the local laboratory of each party but no global order or space-time structure is assumed.[3] The global behaviour is characterised by a *process matrix*, which models the "outside environment" of these local labs (which they cannot access) and encodes information about how the local labs interact. We now review the formal definitions of local operations and process matrices. We will only consider the bipartite case here; multipartite settings are discussed for example in [11].

Local behaviour: local quantum laboratories. Each party acts in a *local quantum laboratory* associated with the input Hilbert space $\mathcal{H}_{A_I}$ of dimension d_{A_I} and output Hilbert space $\mathcal{H}_{A_O}$ of dimension d_{A_O}.[4] The operations performed by agents in their local labs are described by *quantum instruments* $\mathcal{J}^A = \{\mathcal{M}_x^A\}_{x=1}^m$ with $\mathcal{M}_x^A : A_I \to A_O$ [1,12]. Here, x parametrizes the local measurement outcomes, and A_I and A_O represent the set of all Hermitian, linear operators over $\mathcal{H}_{A_I}$ and $\mathcal{H}_{A_O}$ respectively. When the classical measurement setting a is used to characterise the operations, the corresponding instrument is denoted as $\mathcal{J}_a^A = \{\mathcal{M}_{x|a}^A\}_{x=1}^m$. Quantum instruments being a set of CP maps have a corresponding Choi-Jamiołkowski representation [12], and a quantum instrument $\mathcal{J}^A = \{\mathcal{M}_x^A\}_{x=1}^m$ can be equivalently represented by the set of Choi-Jamiołkowski states $\{M_x^{A_I A_O}\}_{x=1}^m$.

Global behaviour: process matrices. The probability $P(x_1, ..., x_N|a_1, ..., A_N)$ that the N agents $\{A^i\}_i$ observe the outcomes $(x_1, ..., x_N)$ for a choice of measurement settings $(a_1, ..., a_N)$ is a function of the corresponding local maps $\mathcal{M}_{x_1|a_1}^{A^1}, ..., \mathcal{M}_{x_N|a_N}^{A^N}$ and a global behaviour, called the process matrix. This can be expressed using the Choi-Jamiołkowski representation of the maps as follows [1,12],

$$P(x_1, ..., x_N|a_1, ..., a_N) = P\left(\mathcal{M}_{x_1|a_1}^{A^1}, ..., \mathcal{M}_{x_N|a_N}^{A^N}\right) = \mathrm{tr}\left[\left(M_{x_1|a_1}^{A_I^1 A_O^1} \otimes ... \otimes M_{x_N|a_N}^{A_I^N A_O^N}\right) W\right], \tag{3}$$

for a Hermitian operator $W \in A_I^1 \otimes A_O^1 \otimes ... \otimes A_I^N \otimes A_O^N$, known as the *process matrix*. The above equation plays the role of the Born rule in these indefinite space-time settings, and W the role of the density matrix. The set of valid process matrices is characterised by the set of all such Hermitian operators that yield positive normalised probabilities for all possible operations, including ancillas, tracing out subsystems and sharing entangled states between multiple agents. This imposes certain conditions on W, such as non-negativity [12].

[2] Without loss of generality, it can be defined with one input wire and one output wire, since multiple wires can be combined into a single wire through the isomorphism of Equation (1).

[3] In other words, there is no fixed metric or global partial ordering $\mathcal{T}$, in contrast to the causal box framework.

[4] These Hilbert spaces are assumed to be finite dimensional.

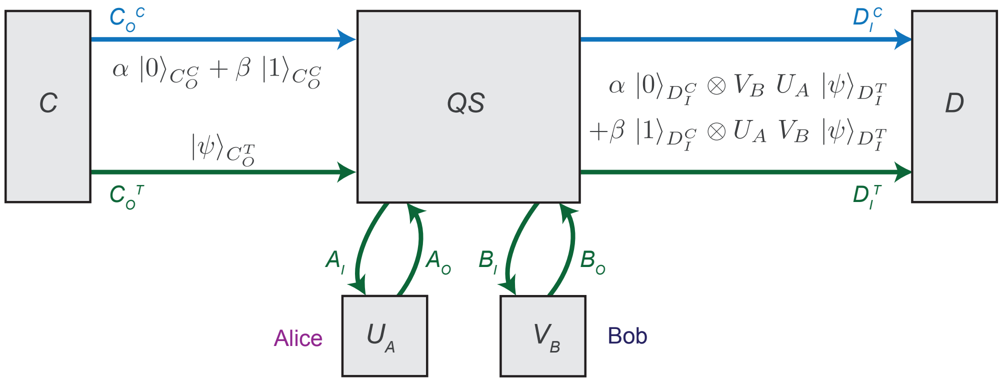

Figure 1: **The quantum switch as a causal box [5]:** Taking the set $\mathcal{T}$ to be $\{1,2,3,4,5,6\}$, at time $t = 1$, the input state of the control and target are received on wires C_O^C and C_O^T respectively. $QS_{t=1}$ moves the control qubit to an internal quantum memory Q and forwards the target from C_O^T into either U_A or V_B depending coherently on the value of the control (and forwards the vacuum state to the other box). At $t = 2$, the unitaries are applied, and at $t = 3$, $QS_{t=3}$ forwards the state received from U_A to V_B and that received from V_B to U_A. At $t = 4$, the unitaries are again applied. At $t = 5$, $QS_{t=5}$ outputs either the state received from U_A or V_B depending coherently on the control qubit stored in the quantum memory Q, it also outputs the control qubit as it is. At $t = 6$, the final outputs are obtained on wires D_I^C (control) and D_I^T (target). Even though the unitaries are applied at two distinct times $t = 2$ and $t = 4$, they are each queried only once in the whole process. The causal box description does not assign these operations to different parties/local labs, and can be treated as a protocol that happens within a single lab (akin to the physical implementation). However, we have used notation analogous to the process matrix description in order to facilitate comparison. Then this can be regarded as a 4 party protocol between A, B, C and D where the subscript I in the wire label denotes that it is an input wire, and the subscript O denotes output wires.

Causality. Process matrices are classified as causal or non-causal depending on whether or not they produce probability distributions compatible with a fixed causal order (or a convex combination thereof). Consider a bipartite scenario with parties Alice and Bob, inputs a and b, and outputs x and y respectively. We say that $A \prec B$ if Bob cannot signal to Alice; the corresponding distribution produced in this case, denoted as $p^{A \prec B}$, would satisfy $p^{A \prec B}(x|a,b) = p^{A \prec B}(x|a,b') \ \forall a,b,b',x$, where $p^{A \prec B}(x|a,b)$ denotes the marginal $\sum_y p^{A \prec B}(x,y|a,b)$. The analogous condition holds for the distribution $p^{B \prec A}$ compatible with the causal order $B \prec A$, where Alice cannot signal to Bob. More generally, in this scenario, a bipartite distribution $p(x,y|a,b)$ is called *causal* if it can be written as a convex combination of ordered distributions i.e.,

$$p(x,y|a,b) = q\ p^{A \prec B}(x,y|a,b) + (1-q)\ p^{B \prec A}(x,y|a,b), \tag{4}$$

for some probability $q \in [0,1]$.[5] The set of all distributions $p(x,y|a,b)$ satisfying Equation (4) form a convex polytope, known as the *causal polytope* [13]. A bipartite process matrix $W_{A_I A_O B_I B_O}$ that produces causal correlations $p(x,y|a,b)$ for all choices of local instruments $\{M^A_{x|a}\}_{x,a}$ and $\{M^B_{y|b}\}_{y,b}$ is called a *bipartite causal process* [11, 13]. Causal inequalities are linear constraints satisfied by all causal distributions, in the same way that Bell inequalities are satisfied for distributions compatible with local strategies.

2.3 The quantum switch

The quantum switch is originally defined as a supermap QS, or higher-order transformation, that acts on the space of channels (which are themselves linear maps) [2]. In particular, given two unitary channels U_A and V_B that act on a target qubit, the quantum switch maps them to the channel that implements the following transformation on a control (C) and target (T) qubits:

$$(\alpha\,|0\rangle + \beta\,|1\rangle)_C \otimes |\Psi\rangle_T \quad \longrightarrow \quad \alpha\,|0\rangle_C \otimes (V_B\ U_A\,|\Psi\rangle)_T \ + \ \beta\,|1\rangle_C \otimes (U_A\ V_B\,|\Psi\rangle)_T\,, \tag{5}$$

where $|\psi\rangle_T$ is an arbitrary pure state of the target. More generally, one can consider the quantum switch operation on non-unitary local instruments. In Figures 1 and 2, we review the causal box and process matrix descriptions of the quantum switch. The causal box framework views the quantum switch as a system or box (which can be composed with other systems such as U_A and V_B, therefore putting them

[5] Note that non-signalling correlations $p(x,y|a,b)$ neither allow signalling from Bob to Alice nor from Alice to Bob and are compatible with both the orders $A \prec B$ and $B \prec A$. Hence the decomposition of Equation (4) is in general not unique.

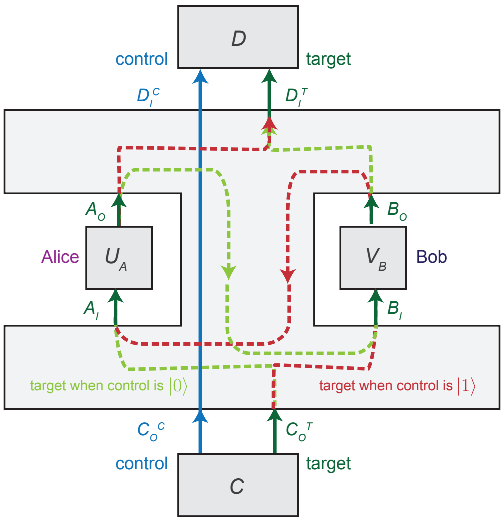

Figure 2: **4 party process matrix for the quantum switch:** A lab C in the past of all others (with trivial input space) prepares the control and target states and sends the target to A if the control is in state $|0\rangle$ and to B if the state is $|1\rangle$. After A and B have operated on the target in an order depending on the control state, a lab D in the future of all others (with trivial output space) receives the target from A or B and control directly from C (blue path). D therefore holds the final state of the control and target where the order of A's and B's operation on the target is entangled with the control state. The process matrix, W for the quantum switch in this case represents a controlled superposition of the orders $C \prec A \prec B \prec D$ (green path) and $C \prec B \prec A \prec D$ (red path).

on equal footing) while the process matrix framework views it as a process modelled by a process matrix (i.e., distinct in nature from the local operations U_A and V_B). These views relate to the particular physical implementations (definite or indefinite space-time) of the quantum switch, as we will discuss later. In the following, we will refer to the quantum switch as a transformation, a description that subsumes both these view points.

3 Results

The main goal of this paper is to compare the causal box and process matrix frameworks and establish results connecting them, which would provide insights into indefinite causal orders, causal inequalities and their physical realisations. A priory, the frameworks are very different and it is not obvious which objects can being compared:

- The process matrix framework describes separate local labs with a local order of events, quantum states entering and exiting each lab only once, and a process matrix describing the outside environment of the labs which cannot be accessed by local agents and may not admit a fixed background space-time.
- The causal box framework makes no such separation into local labs: it considers systems with input and output wires which can be composed to produce other systems, against a background of a fixed space-time. It allows for outputs to be looped back into later inputs, multiple quantum messages (or a superposition thereof) to enter and exit the boxes, and generalised operations that can act non-trivially on both the quantum message and space-time location associated with it.

Therefore we need to introduce a language in which such a comparison can be made, by imposing certain constraints on causal boxes that allow them to act as "processes" (higher-order maps) on local operations. We call such causal boxes *process boxes*.

3.1 Process Boxes: a formalism for comparing systems and processes

Local behaviour: local lab/operation boxes. We model each local lab as a causal box. However, in order to be comparable to the process matrix framework and to avoid trivial causal inequality violations using, certain restrictions must be imposed on these boxes. These restrictions mathematically model the following assumptions that are implicit in the PM framework:

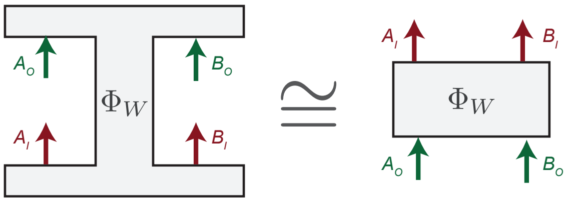

Figure 3: **A bipartite process box.** A bipartite process box is a causal box with input wires A_O and B_O and output wires A_I and B_I that effectively acts as a higher order map on causal boxes representing local operations of A and B that satisfy the WSR, OSR and LO conditions described in Section 3.1.

1. The input and output wire spaces of these boxes must be restricted to allow only for at most one message to be received and sent (and not multiple messages at different times).

2. The input to each lab must precede the output, even when the inputs and outputs to each local lab may be received or sent at a superposition of different times.

3. The local operations are time-translation invariant i.e. their action on the classical/quantum message is independent of the space-time location at which the message is sent/received.

Conditions 1 and 2 are required to rule out trivial causal inequality violation using fixed causal orders. For example, if either 1 or 2 were not satisfied in a bipartite scenario, two-way communication between the parties would be allowed and they could trivially produce non-causal distributions that cannot be expressed in the form of Equation 4, by exchanging their classical inputs (see Figure 4). Condition 3 is more subtle: firstly, it facilitates a comparison with the PM framework which does not consider space-time labels explicitly as the CB framework does, and the local operations act only on the quantum states. Secondly, the time stamps on quantum messages may carry information regarding the order in which the parties act and if the local operations leak any information about the time at which they acted on a non-vacuum state, this could decohere superpositions of such orders. For example, in the optical realisation of the quantum switch (and the CB description of Figure 1), the time at which a non-vacuum state enters each lab depends entirely on the value of the control and measuring this time would collapse the superposition.[6]

More formally, an N-partite scenario with local labs $A^1, ..., A^N$ is modelled by N causal boxes $\mathcal{M}_{A^1}$, $\ldots$, $\mathcal{M}_{A^N}$ describing the local operation performed in each lab. Each box $\mathcal{M}_{A^i}$ has an input wire A_I^i and an output wire A_O^i (of dimensions d_I^i and d_O^i respectively), and corresponding state spaces $\mathcal{A}_I^i$ and $\mathcal{A}_O^i$. Measurements are modelled analogously to the process framework (Section 2.2) where for each lab A, $\mathcal{M}^A$ plays the role of the quantum instrument and is modelled by a set of maps $\{\mathcal{M}_x^A\}_{x=1}^m$, one for each measurement outcome x. And if this additionally depends on a classical input a (measurement setting), then we analogously have the maps $\{\mathcal{M}_{x|a}^A\}_{x=1}^m$ for each choice of a. The crucial difference is that these maps act on different quantum state-spaces (which include the vacuum and time stamps) as compared to those of the process matrix framework. These state spaces are defined more precisely through the restrictions enumerated below. For the current purposes, we can assume each local lab to be fixed at a given spatial location (such that the spatial location directly corresponds to the lab label) and take the set $\mathcal{T}$ to be totally ordered, representing time in a global, shared reference frame.[7] Then the lab label along with the time information in $\mathcal{T}$ give the space-time location of each message. The assumption of a global reference frame does not limit our results in any way, but only simplifies the statements, because these are based on the partial order induced by causal pasts and futures which are frame independent notions. Then, we find that the above mentioned restrictions can be modelled by the following mathematical conditions:

1. *Wire-space restriction (WSR):* The input (and output) wire spaces are associated with the restricted Fock spaces that allow only for zero or one message.

$$\mathcal{F}_{A_I^i}^{\mathcal{T}} = \bigoplus_{n=0}^{1} \mathcal{H}_{A_I^i} = (|\Omega\rangle \oplus \mathbb{C}^{d_I^i}) \otimes l^2(\mathcal{T}) = \mathcal{H}_{A_I^i}^Q \otimes \mathcal{H}_{A_I^i}^T, \tag{6}$$

and analogously $\mathcal{F}_{A_O^i}^{\mathcal{T}}$ for the output. The spaces $\mathcal{H}_{A_I^i}^Q$ and $\mathcal{H}_{A_I^i}^T$ correspond to the Hilbert spaces of the

[6] Considering a relaxation of this condition (which cannot be modelled in the PM framework) would be an interesting exercise for future work. It is a priory not clear whether such a relaxation would leave us better or worse off when it comes to violating causal inequalities, since on the one hand they can decohere superpositions of orders, while on the other, they can allow for a larger class of local operations that include non-trivial operations on quantum messages and their space-time stamps.

[7] For future work, one may consider scenarios where there is no shared reference frame (but only local reference frames) or where the parties may be in relative motion.

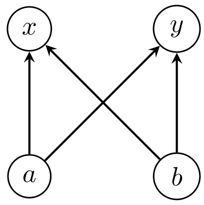

Figure 4: **A fixed causal structure that can lead to causal inequality violations** Here a and b are classical inputs, and x and y are classical outputs of Alice and Bob respectively. Clearly, if Alice and Bob are allowed to exchange their inputs in this manner, they can trivially generate non-causal distributions that don't satisfy Equation (4). But note that this would either require at least one party to send an output before they receive an input (violating local order) or send/receive more than one message (violating wire-space restriction).

message and that of the time stamp respectively.[8] Then the input and output state spaces for each local lab are $\mathcal{A}_I^i := \mathfrak{T}(\mathcal{F}^{\mathcal{T}}_{A_I^i})$ and $\mathcal{A}_O^i := \mathfrak{T}(\mathcal{F}^{\mathcal{T}}_{A_O^i})$, where $\mathfrak{T}(\mathcal{F})$ denotes the set of all trace class operators on the space $\mathcal{F}$.

2. *Operation-space restriction (OSR):* Each local lab A^i performs a time-translation invariant operation. More formally, the local operation maps an input received at time t to an output at time $t+\Delta$ for some $\Delta > 0$ independent of the time t [9] and for each i, the map $\mathcal{M}_{A^i} : \mathcal{A}_I^i \mapsto \mathcal{A}_O^i$ can be decomposed as $\mathcal{M}^Q_{A^i} \otimes \Phi^T_{A^i}$ which act on the spaces $\mathcal{H}^Q_{A_I^i}$ and $\mathcal{H}^T_{A_I^i}$ respectively. Here $\Phi^T_{A^i}$ simply maps the time stamp $|t\rangle$ to $|t+\Delta\rangle$ $\forall t \in \mathcal{T}$.

3. *Local order (LO):* For each Lab A^i with non-trivial input i.e., $d_I^i > 1$, the map $\mathcal{M}_{A^i} : \mathcal{A}_I^i \mapsto \mathcal{A}_O^i$ leaves the vacuum state $|\Omega\rangle$ invariant, i.e., $\mathcal{M}_{A^i}|\Omega, t\rangle = |\Omega, t+\Delta\rangle$ for all times t, where Δ is as defined above. This ensures that any non-vacuum output must be preceded by a non-vacuum input, thereby imposing the local order requirement in a coherent manner, without having to measure the local time to order these events.

Remark 1. *Another assumption required for ruling out trivial causal inequality violations is the closed lab assumption [1]. Essentially, this requires that the only way that a local lab A interacts with anything outside it, is through its quantum input A_I and output A_O, and the lab is "closed" to another inputs/outputs at all other times. This is implicit in the PM framework as it is in our current formalism, which is what allows us to ignore time-stamps on the classical input and output to each lab. For example, each local operation $\mathcal{M}^A$ here can be thought of as a causal box with input wires A_I and a, output wires A_O and x. Here A_I and A_O may carry quantum messages and each of the wires a and x only carry a single classical message corresponding to the wire name. The time-stamps on these classical messages are not relevant for the current purpose because any information about these can only exit the lab through the output A_O (whose time stamp is taken into account), the input a is assumed to be chosen freely, which is why we simply treat the a and x as labels in analogy with the process framework. However, these time-stamps could become relevant in physical realisations where one may not want these variables to be generated too far in the past of an experiment, in order to justify free choice or the closed lab assumptions.*[10]

Global behaviour: process box An N-partite process box Φ_W^N is a causal box with input wires A_O^1,...,A_O^N and output wires A_I^1,...,A_I^N. Wires in the CB framework can in general carry several messages at different times as well as superpositions of different number of messages,. However, under the restrictions identified above, a process box effectively acts as a higher order map on the local operations $\{M^{A^i}\}_i$ (each viewed as a causal box with inputs A_I^i and a_i and outputs A_O^i and x_i, as explained in Remark 1) of each party and its action can be restricted to the state spaces $\mathcal{A}_I^i$ and $\mathcal{A}_O^i$ ($i \in \{1,...,N\}$) defined above which carry at most one message each. A bipartite process box Φ_W^2 is illustrated in Figure 3. More generally, an N-partite process box Φ_W^N acts on local operations by mapping them to a valid probability distribution $P(x_1,...,x_N|a_1,...,a_N)$ through the composition operation on the causal boxes Φ_W^N, M^{A^1},..., M^{A^N} as explained in Figure 5.

[8] Note that the symmetrization is not required since the symmetric subspaces of $\mathcal{H}^{\otimes 0}$ and $\mathcal{H}^{\otimes 1}$ are the spaces themselves.

[9] The causal box framework does not allow the boxes to act instantaneously, a minimum non-zero operation time is required to satisfy the causality definition.

[10] Modelling these assumptions for physical scenarios in the presence of delocalised events and indefinite space-time would be an interesting future work.

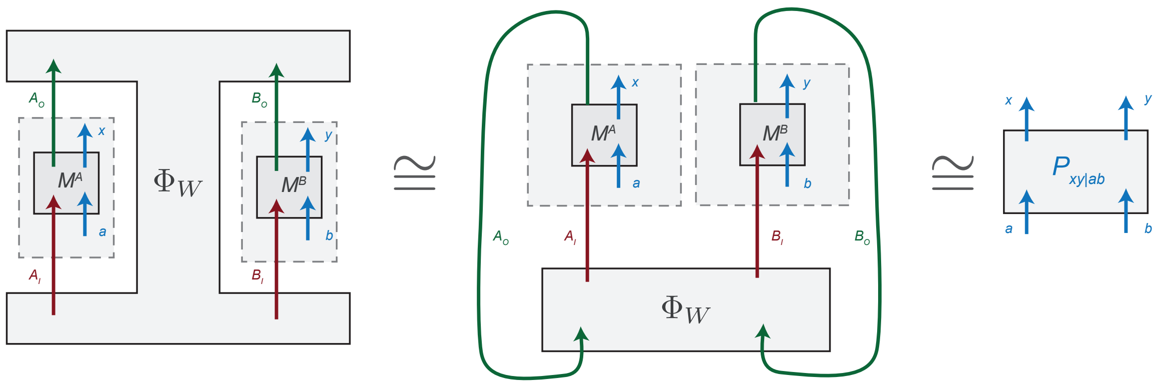

Figure 5: **A bipartite process box composed with local operations** Since causal boxes are closed under composition, the composition of a bipartite process box Φ_W with the local operations M_A and M_B yields a new causal box with classical input wires a, b and classical output wires x and y as shown. The wire-space and operation-space restrictions along with the local order assumption imply that the resulting classical causal box can be described by a single probability distribution $P_{xy|ab}$. We show that this can be obtained by an analogous formula to Equation (3) with the (Choi-Jamiolkowski representations of the) process box and local operation boxes replacing the process matrix and local operations respectively.

3.2 Representations of QS

Having established a framework within which a comparison of causal boxes and process matrices can be made, we first compare the representation of the quantum switch in the two frameworks since this is an example of an indefinite causal/temporal order that can be modelled in both frameworks. We consider the quantum switch as a transformation involving 4 parties A, B, C and D where C is in the past of all others and D in the future of all others. The description of this transformation in the CB and PM frameworks was presented in Section 2.3, where the dimensions of the inputs and outputs to the labs/local boxes are as follows: $d_I^A = d_O^A = d_I^B = d_O^B = 2$, $d_I^C = d_O^D = 1$ and $d_O^C = d_I^D = 4$. Further, the 4 dimensional output wire of C and input wire of D are split into two 2 dimensional wires C_O^C and C_O^T (and analogously for D_I) for the control and target qubits respectively. In Theorem 1, we compare the causal box and process matrix descriptions of QS based on the assumptions that we have identified and we find that the representations of QS in the two frameworks are *not* equivalent. We require the following concepts for stating this Theorem.

Ignoring vacuum states: When modelling scenarios where each of the parties involved operate upon a non-vacuum state exactly once, one can ignore all channels carrying vacuum states while preserving the meaning of the process. This can be done using the isomorphism of Equation (1) where a tensor product of two vacuum states on the right hand side of the equation is mapped to a vacuum state on the left, a tensor product of a vacuum state and one message on the right is mapped to a single message with the same value and position on the left, etc. For example, $\mathrm{Span}\Big(\{|0,t_1\rangle|\Omega,t_2\rangle, |1,t_1\rangle|\Omega,t_2\rangle, |\Omega,t_1\rangle|0,t_2\rangle, |\Omega,t_1\rangle|1,t_2\rangle\}\Big) \cong \mathrm{Span}\Big(\{|0,t_1\rangle, |1,t_1\rangle, |0,t_2\rangle, |1,t_2\rangle\}\Big)$ under this isomorphism. For the Choi state, this would correspond to ignoring channels that track the vacuum state. We refer the reader to [5] for a full characterisation of this isomorphism.

Ignoring time-stamps: In the causal box QS (Figure 1), the parties A and B receive a non-vacuum state at an earlier time $t = 2$ and a vacuum state at a later time $t = 4$ or vice-versa depending on the control qubit. Ignoring the time-stamps corresponds to dropping them from the states/wire labels (e.g., $A_I^{t=2}, A_I^{t=4} \mapsto A_I$). This makes sense since for each value of the control, only one of these time-stamped wires would carry a non-vacuum state. Note that ignoring time-stamps is not a valid operation within the CB framework, but one that we propose for mapping the CB description to the PM one under the identified assumptions. Finally, his allows us to ignore the vacuum states appearing in terms such as $\sum_{j\in\{\Omega,0,1,..\}}|j\rangle|j\rangle$ $\sum_{j\in\{\Omega,0,1,..\}}|j\rangle\langle j|$ since the vacuum state does not appear in the basis of the associated state-spaces in the PM framework.[11]

[11]One can view this as moving from a time-delocalised to a time-localised description of the local operations. The latter corresponds to the gravitational QS [9] where a non-vacuum state enters A/B's lab at a fixed local time, independent of the value of the control, and the vacuum state also decouples from the process.

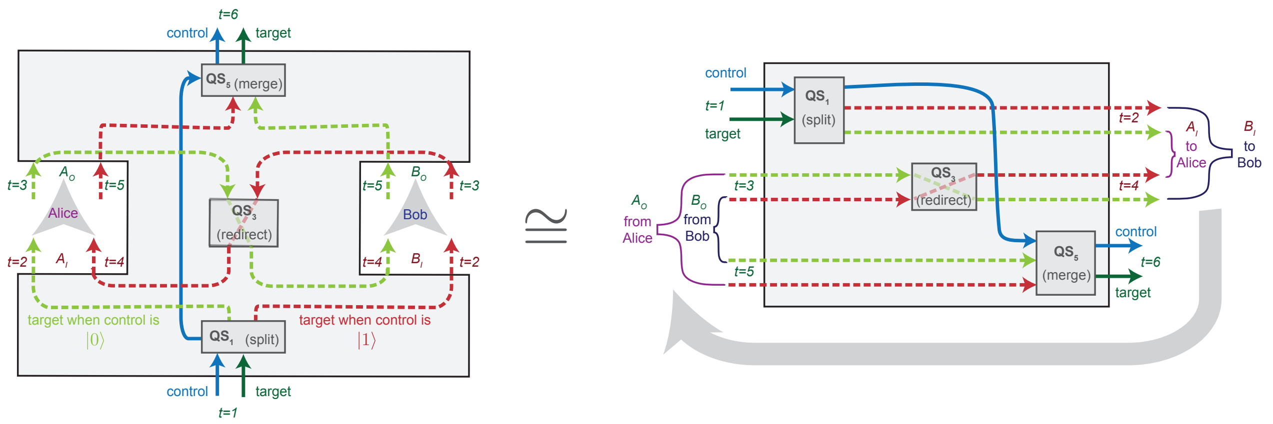

Figure 6: **Sequence representation of causal box QS depicted in Figure 1 (right) and that of the equivalent process box (left)**

Theorem 1 (Quantum switch in both representations)**.** *If the causal boxes $\mathcal{M}_A$, $\mathcal{M}_B$, $\mathcal{M}_C$ and $\mathcal{M}_D$ representing the local operations of A, B, C and D satisfy the WSR, OSR and LO conditions (Section 3.1), then the effective CJ representation of the causal box QS (under these assumptions) maps to the process matrix W_{QS} when all vacuum states and time information are ignored.*

Explanation The causal box QS can be seen as a map from input wires $C_O^{C,t=1}$, $C_O^{T,t=1}$, $A_O^{t=3}$, $B_O^{t=3}$, $A_O^{t=5}$, $B_O^{t=5}$ to output wires $D_I^{C,t=1}$, $D_I^{T,t=1}$, $A_I^{t=2}$, $B_I^{t=2}$, $A_I^{t=4}$, $B_I^{t=4}$ (See Figure 6 and 7). We find if QS is composed with local operations $\mathcal{M}_A$, $\mathcal{M}_B$, $\mathcal{M}_C$ and $\mathcal{M}_D$ that satisfy the WSR, OSR and LO conditions of Section 3.1, then the causal box QS can be *effectively* described by the state (writing $C_O^{t=1}$ and $D_I^{t=6}$ without the time-labels for brevity)

$$\begin{aligned}|\phi_{QS}\rangle = &|00\rangle^{C_O^C D_I^C} |\mathbb{1}\rangle\!\rangle^{C_O^T A_I^{t=2}} |\mathbb{1}\rangle\!\rangle^{A_O^{t=3} B_I^{t=4}} |\mathbb{1}\rangle\!\rangle^{B_O^{t=5} D_I^T} |\Omega\rangle^{B_I^{t=2}} |\Omega\Omega\rangle^{B_O^{t=3} A_I^{t=4}} |\Omega\rangle^{A_O^{t=5}} \\ &+ |11\rangle^{C_O^C D_I^C} |\mathbb{1}\rangle\!\rangle^{C_O^T B_I^{t=2}} |\mathbb{1}\rangle\!\rangle^{B_O^{t=3} A_I^{t=4}} |\mathbb{1}\rangle\!\rangle^{A_O^{t=5} D_I^T} |\Omega\rangle^{A_I^{t=2}} |\Omega\Omega\rangle^{A_O^{t=3} B_I^{t=4}} |\Omega\rangle^{B_O^{t=5}},\end{aligned} \tag{7}$$

where $|\mathbb{1}\rangle\!\rangle = \sum_{j\in\{\Omega,0,1\}} |j\rangle|j\rangle$. Note that this is only an *effective* description of the causal box QS and not its complete description which would include terms corresponding to multiple messages on the wires, which are ruled out by the assumptions.[12] Then ignoring the vacuum states $|\Omega\rangle$ and ignoring time stamps, gives the process vector

$$|W_{QS}\rangle = |00\rangle^{C_O^C D_I^C} |\mathbb{1}\rangle\!\rangle^{C_O^T A_I} |\mathbb{1}\rangle\!\rangle^{A_O B_I} |\mathbb{1}\rangle\!\rangle^{B_O D_I^T} + |11\rangle^{C_O^C D_I^C} |\mathbb{1}\rangle\!\rangle^{C_O^T B_I} |\mathbb{1}\rangle\!\rangle^{B_O A_I} |\mathbb{1}\rangle\!\rangle^{A_O D_I^T}, \tag{8}$$

where $|\mathbb{1}\rangle\!\rangle$ now corresponds to the sum without the vacuum states $\sum_{j\in\{0,1\}} |j\rangle|j\rangle$.

If either the wire-space restriction or the local order condition did not hold, the causal box QS could be used to trivially violate causal inequalities by exchanging classical information between Alice and Bob as shown in Figure 4. Theorem 1 shows that if these restrictions are imposed, the causal box QS can be mapped to process matrix $W_{QS} = |W_{QS}\rangle\langle W_{QS}|$ which does violate any causal inequalities [12]. Note however that this map is irreversible since it involves ignoring the time stamps and it is possible to have a different assignment of local time-stamps for the same process matrix which may not be described by a causal box. The indefinite space-time realisation of QS would one such example where the local-time stamps are fixed irrespective of the order, and this cannot be modelled completely using causal boxes where a fixed background partial order is hard coded in to the framework.

3.3 Process boxes and causal inequalities

Theorem 2. *All bipartite process boxes Φ_W^2 between parties A and B, when composed as in Figure 5 with local operations M^A and M^B that satisfy the WSR, OSR and LO conditions, yield probability distributions $P_{xy|ab}$ that are causal (Equation (4)) and hence cannot violate causal inequalities.*

[12] These terms do not contribute to the measurement statistics when composed with local operations satisfying the OSR, WSR and LO assumptions but when composed with operations violating these, can lead to trivial causal inequality violations as explained in Figure 4.

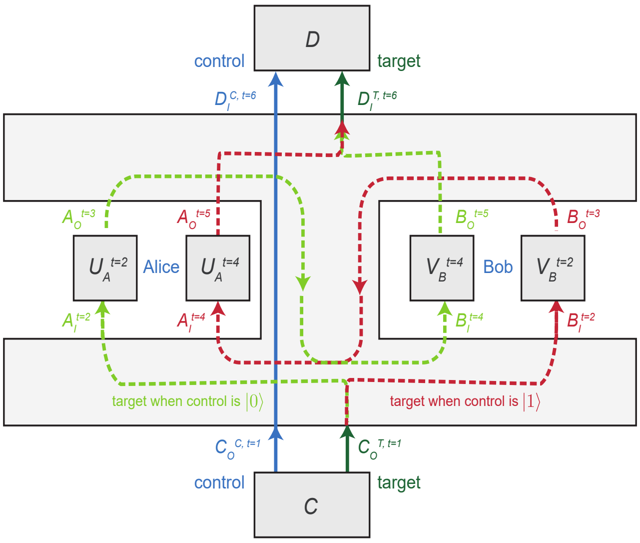

Figure 7: **Sequence representation of QS composed with local operations** The comparison of this figure with the process matrix representation of Figure 2 illustrates the intuition behind Theorem 1. Here, the green path tracks the path of the non-vacuum target state when the control is $|0\rangle$ and the red tracks the path of this state when the control is $|1\rangle$. Note that the vacuum state follows the opposite path in each case (red when control is $|0\rangle$ and green when it is $|1\rangle$) but this is not part of the process matrix description. The wire-space restriction is implicit in the definition of the input/output spaces. The operation-space restriction and local order conditions are illustrated through the splitting of the local operations of A and B into those at two separate times such that each of them map an input at t to an output at $t+1$ without otherwise acting on the time-stamps. Ignoring the time stamps would correspond to combining the input/output wires and operations U_A and V_B at the distinct times into a single wire operation. When this is done, one can see that the current Figure would resemble that of the corresponding process matrix (Figure 2) i.e., for local operations U_A and V_B, preparation at C and measurement at D satisfying wire-space and operation space restrictions along with the local order condition, the CB representation maps to the PM one when vacuum and time information are ignored (Theorem 1).

The intuition for Theorem 2 is the following. Firstly, causal inequalities cannot be violated by convex mixtures of one-way signalling scenarios, since these would only produce causal distributions (Equation 4). In order to have two way communication without violating the causality condition for causal boxes, Alice's output at time t_{A_O} must be fed into Bob's input at time $t_{B_I} > t_{A_O}$ and Bob output at t_{B_O} must be fed into Alice's input at $t_{A_I} > t_{B_O}$. But the local order condition requires that $t_{A_I} < t_{A_O}$ and $t_{B_I} < t_{B_O}$ which cannot simultaneously be satisfied without leading to a time-like loop that violates the causality condition of the CB framework. If we additionally provide a non-vacuum input at t_0 to Alice(/Bob) before $t_{A_I}(/t_{B_I})$ to "activate" the output at $t_{A_O}(/t_{B_O})$ according to the local order assumption, then the wire-space restriction would fail since the lab would be receiving non-vacuum states at t_0 and $t_{A_I}(/t_{B_I})$ which constitute two 2-dimensional messages on the 2-D input wire. Further, we find that no pure superposition of these channels is possible in the bipartite case since these would implement uncontrolled superpositions of orders which do not correspond to valid maps[13]. Hence, only convex combinations of causally ordered processes can be realised this way, which by construction don't violate causal inequalities.

It is known that all bipartite unitary extendable processes are causal [14]. This is in agreement with Theorem 2 for process boxes, which we find to also be unitary extendable. In the multi-partite case, there are unitary extendable processes that are non-causal [15], and are known to have realisations in Oreshkov's time-delocalised systems framework [6,16]. However, realisations of such processes require a cyclic structure of channels carrying non-vacuum states, which may not be implementable using causal boxes that satisfy stricter causality requirements (the existence of a sequence representation for example [5]) than that of Oreshkov's framework. Hence, we believe that the intuition behind our proof for the bipartite case would still apply to these general scenarios: causality along with the assumptions identified in this paper would forbid sequences of channels between parties that form loops (even if these loops lead to valid probabilities)[14],

[13]In the case of QS, there is a controlled superposition of the orders and the orthogonal states of the control ensures orthogonality between the 2 orders being superposed which is not possible without a control.

[14]Even though QS is shown to have a a cyclic causal structure in [14], the channel structure of the corresponding causal box does not have any loops due to the different time stamps.

imposing an underlying partial order on the channel structure. Further, superpositions of orders without a control could be ruled out in a similar manner[15]. This suggests that the only indefinite causal/temporal orders implementable using causal boxes are those that correspond to controlled (and possibly dynamic) superpositions such as QS, which can be shown to be causal [12]. Hence we conjecture the same and leave the proof for future work.

4 Discussion

Definite and indefinite space-time realisations of QS Theorem 1 points to important distinctions between the optical (definite space-time) and gravitational (indefinite space-time) realisations of QS. While both share the same process matrix, only the former can be modelled as a causal box[16]. This explains the irreversibility of the map from the CB to PM representations of QS; in going from the PM to CB description, time stamps must be assigned and this assignment is not unique. The CB representation corresponds to a particular assignment of time-stamps which are correlated with the control and allow it to admit a causal unravelling within a fixed space-time while the gravitational case corresponds to a different assignment of time-stamps, where the local time at which a non-vacuum target state enters each local lab is independent of the value of the control. This suggests that the definite space-time realisation merely implements a superposition of temporal order of operations, while the indefinite space-time version implements a superposition of causal orders since space-time geometry which defines causal order is itself in superposition (see also [17]).

Vacuum states The importance of vacuum states in physical tasks is best demonstrated by the task of controlling a black-box unitary, which was shown to be impossible in theory [18, 19] yet implemented in practice [20]! This is essentially because the theoretical proof did not consider vacuum states while the experiment exploited the ability to send a superposition of "something" and "nothing" to a black-box. Further, the vacuum state (which represents "nothing") can also carry information: for example, in the quantum switch, if Alice receives nothing (i.e., the vacuum state) at the earlier time ($t = 2$), she would know that Bob received the target state before her, thereby inferring the value of the control qubit (even though this was never physically sent to her), and collapsing the superposition. Note that both vacuum states and time-stamps are central to such examples.

Assumptions behind causal inequalities We presented the assumptions for ruling out trivial causal inequality violations in the CB framework. However a framework independent formulation of these assumptions that would apply to both definite and indefinite space-times, as well as space-time delocalised events, is still lacking. It is unclear whether this can be done without considering vacuum states and local time stamps (specially for the local order condition). Typically, for superpositions such as QS, the wire-space restriction is imposed by having a coherent counter at each local lab and measuring it at the end to confirm that only one message (of a fixed dimension) entered and exited the lab during the process. But this requires the counters to decouple from from everything else at some point that one could call the "end" of the process. This holds in the case of QS [5] but cannot be easily defined/imposed in general. For example, in a protocol where a target qubit goes to Alice and then to Bob if the control qubit is $|0\rangle$ and only to Bob if the control is $|1\rangle$, Alice's counter would be entangled with the control and cannot be measured at any point without disturbing other states involved. In fact, an important question would be whether these assumptions can be formulated in a device-independent way, without referring to the structure of states and operations, analogous to Bell inequalities.

5 Conclusions

There exist several frameworks for modelling causality in quantum and more general theories that are applicable within different regimes or under different sets of assumptions. Comparing different frameworks and establishing connections between them are important steps towards a more complete understanding of causality in different regimes, including those where quantum as well as gravitational effects play a role. This will provide crucial insights into which causal phenomena may be physically realisable (and why), whether these phenomena can be advantageous for information processing tasks, and possibly also provide

[15]Based on personal correspondence, we believe that a similar result establishing impossibility of pure processes implementing such superpositions has been shown in unpublished work by Fabio Costa [private correspondence].

[16]This is because the notion of a fixed space-time $\mathcal{T}$ is hard coded into the framework, and indefinite space-time geometries resulting from superpositions of gravitating masses cannot be completely modelled by a single partial order $\mathcal{T}$.

some insights into quantum gravity. To the effect, we studied the process matrix (PM) and causal box (CB) frameworks. The former can model indefinite space-times and causal structures, and describe processes or higher order maps on local operations, but ignore vacuum states and time information that are crucial in physical implementations. The latter consider vacuum states and time labels but models systems embedded in a definite space-time, that can repeatedly exchange information and act as maps on states. We develop a language for comparing these frameworks by identifying assumptions required to rule out trivial causal inequality violations in CBs. This allows us to treat causal boxes as local operations on states and also as higher order transformations on these local operations, depending on the assumptions imposed. Within this formalism, we show that the representation of the quantum switch in the CB and PM frameworks is not equivalent, identifying the map from the CB to the PM representation and showing that it is not reversible. We also show that bipartite causal boxes under the identified assumptions (which we call “process boxes”) cannot violate causal inequalities. Our formalism and initial results allow for further comparison and extension of these frameworks, for example the extension of the CB framework to indefinite space-times or that of the PM framework to model joint local operations on quantum states along with their local time stamps.

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