The Torpedo Game: Quantum Advantage, Wigner Negativity, and Sequential Contextuality in a Generalised Random Access Code

Pierre-Emmanuel Emeriau
Laboratoire d'Informatique de Paris 6
CNRS and Sorbonne Université
pierre-emmanuel.emeriau@lip6.fr

Mark Howard
School of Mathematics, Statistics & Applied Mathematics
National University of Ireland, Galway
mark@markhoward.info

Shane Mansfield
Paris Centre for Quantum Computing
Institut de Recherche en Informatique Fondamentale
University of Paris
smansfield@irif.fr

We situate random access codes within a broader class of information retrieval tasks in communication scenarios and investigate how quantum advantage and contextuality are manifested in this setting. As an example of a task from this more general class, we introduce the Torpedo Game. It is a single-system communication game subject to a restricted set of measurements, which in particular are not sufficient to display contextuality of the Bell–Kochen–Specker type. It also admits a geometric interpretation through which it may be played as a pacifist alternative to the popular game *Battleship*. We show how an operational advantage is achievable by using states with negativity in a discrete Wigner function. By translating the Torpedo Game into a sequence of operations on a fixed preparation, followed by a fixed measurement setting, we show how sequential contextuality furnishes a quantified explanation of where the quantum advantage is derived.

Introduction

Random access coding involves the encoding of a random input string into a shorter message string to be communicated to a second party. The encoding should be such that any element of the original string can be retrieved with high probability from the message. Such tasks have long been studied as examples in which the communication of quantum information can provide *advantage*, i.e. enhanced performance, over classical information (e.g. [5, 35, 37, 26, 38, 15, 3, 22]) despite the Holevo bound [28] which states that n qubits are required to transmit faithfully n bits of classical information.

In this work, we situate random access codes within a broader class of information retrieval tasks in communication scenarios. As an example of a task from this more general class, we introduce the Torpedo Game. It can be viewed as a generalised random access coding task with additional requirements involving the possibility of retrieving *relative* information about elements of the input string. We show that it also admits a neat geometric interpretation via which it can be presented as a pacifist alternative to the popular game *Battleship*. We prove that optimal classical strategies for the Torpedo Games with bit and trit inputs fail to win the game deterministically.

We develop an analysis of the tasks in terms of the Discrete Wigner Function, negativity of which is a signature of non-classicality that has been studied elsewhere as a resource for quantum speed-up and advantage [23, 39, 30, 18, 11, 34, 14]. This analysis leads to quantum strategies with maximal Wigner

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negativity that outperform classical strategies, in the trit-inputs case succeeding deterministically. We highlight that the Torpedo Game for trit-inputs admits a greater quantum advantage than the comparable random access coding task.

Finally, we investigate the *source* of quantum advantage in information retrieval tasks and in particular in the Torpedo game in terms of a non-classical feature in dimensionally-restricted ontologies known as sequential contextuality [32]. This is a distinct feature to *preparation* contextuality [36] that has been linked to QRACs in other works [37, 15, 4], and to the widely-studied notion of contextuality due to Bell [9], Kochen and Specker [31].

We show that strong sequential contextuality is necessary and sufficient for deterministic success in any information retrieval task that does not admit a perfect classical strategy and can be expressed in an operationally sequential form. Moreover we find a quantifiable relationship between the degree of advantage that can be obtained by a given strategy and the degree of sequential contextuality it exhibits.

1 Information Retrieval in Communication Scenarios

1.1 Random Access Codes

An $(n,m)_2$ **Random Access Code (RAC)**, sometimes denoted $n \to m$, is a communication task in which one aims to encode information about a random n-bit input string into an m-bit message where $m < n$, in such a way that any one of the input bits may be retrieved from the message with high probability. An $(n,m)_2$ **Quantum Random Access Code (QRAC)** instead encodes the input into an m-qubit (quantum) message state.

Such tasks may be considered as two-party cooperative games in which the first party, Alice, receives a random input string from a referee, then encodes information about this in a message that is communicated to the second party, Bob. The referee then asks Bob to retrieve the value of the dit at a randomly chosen position in the input string. We will be assuming that the referee's choices are made uniformly at random.

For instance, for the $(2,1)_2$ RAC game [5], an optimal (classical) strategy is for Alice to directly communicate one of the input bits to Bob. If asked for this bit, Bob can always return the correct answer, otherwise Bob guesses and will provide the correct answer with probability $\frac{1}{2}$. Thus the game has a classical value of $\theta^C_{2\to 1} = \frac{1}{2}\left(1+\frac{1}{2}\right) = \frac{3}{4}$. Quantum strategies can outperform this classical bound.

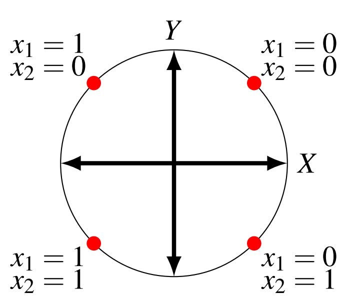

Figure 1: The four red dots correspond to the four states $|\psi_{x_1,x_2}\rangle$ defined in Eq. (1) depicted as points on the equator of the Bloch sphere.

An optimal quantum strategy is for Alice to communicate the qubit state

$$|\psi_{x,z}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + \frac{1}{\sqrt{2}}\left((-1)^x + (-1)^z i\right)|1\rangle\right) \tag{1}$$

where (x,z) is the input bit-string she has received. Bob measures in the X basis when asked for x and in the Y basis when asked for z (see Fig. 1). If he obtains the $+1$ eigenvalue he returns the value 1 and if he obtains the -1 eigenvalue he returns 0. This yields a quantum value for the game of $\theta^{Q}_{2\to 1} = \cos(\frac{\pi}{8})^2$ which is approximately 0.85.

1.2 General Information Retrieval Tasks

In general we may wish to consider a wider variety of communication scenario. In an $(n,m)_d$ **communication scenario** the input is a random string of $d$its and the message is a string of (qu)$d$its, for $d \geq 2$. Within such scenarios one may also consider (Q)RAC tasks like those of Section 1.1. These have been considered elsewhere, e.g. in [38, 13].

We also wish to accommodate for a much wider range of information retrieval tasks. An **information retrieval task** in an $(n,m)_d$ communication scenario is specified by a tuple $\langle Q, \{w_q\}_{q\in Q}\rangle$, where[1]

- Q is a finite set of *questions*,
- the $w_q : \mathbb{Z}_d^n \to \mathbb{Z}_d$ are the *winning relations*, which pick out the good answers to question q given an input string in $\mathbb{Z}_d^n$.

Example Standard $(n,m)_d$ (Q)RACs are recovered when the questions ask precisely for the respective input dits. In that case $Q = \{q_1, \ldots, q_n\}$, and the winning relations $w_i = \pi_i$ are simply projectors onto the respective dits of the input string.

Other interesting tasks will be seen to arise when the questions may concern relative information about the input string, in the form of parities or linear combinations modulo d of the input dits.

2 The Torpedo Game

Of particular interest is an information retrieval task for $(2,1)_d$ communication scenarios, which taking the game perspective we will refer to as the dimension d **Torpedo Game**. Let the input dits be x and z, respectively. There are $d+1$ questions $Q = \{\infty, 0, 1, \ldots, d-1\}$, where the labelling comes from a geometric interpretation that will be elaborated upon shortly. Winning relations for the Torpedo Game are given by

$$\begin{aligned}
w_\infty(x,z) &= \neg x &&= \{a \in \mathbb{Z}_d \mid a \neq x\} \\
w_0(x,z) &= \neg(-z) &&= \{a \in \mathbb{Z}_d \mid a \neq -z\} \\
w_1(x,z) &= \neg(x-z) &&= \{a \in \mathbb{Z}_d \mid a \neq x-z\} \\
w_2(x,z) &= \neg(2x-z) &&= \{a \in \mathbb{Z}_d \mid a \neq 2x-z\} \\
&\vdots \\
w_{d-1}(x,z) &= \neg((d-1)x-z) &&= \{a \in \mathbb{Z}_d \mid a \neq (d-1)x-z\}\,.
\end{aligned} \tag{2}$$

All arithmetic is modulo d.

[1] It is assumed that inputs and outputs are drawn from the commutative ring $\mathbb{Z}_d$. As an aside, we note that information retrieval tasks have the structure of a Chu Space over the two element set with answers in $\mathbb{Z}_d$.

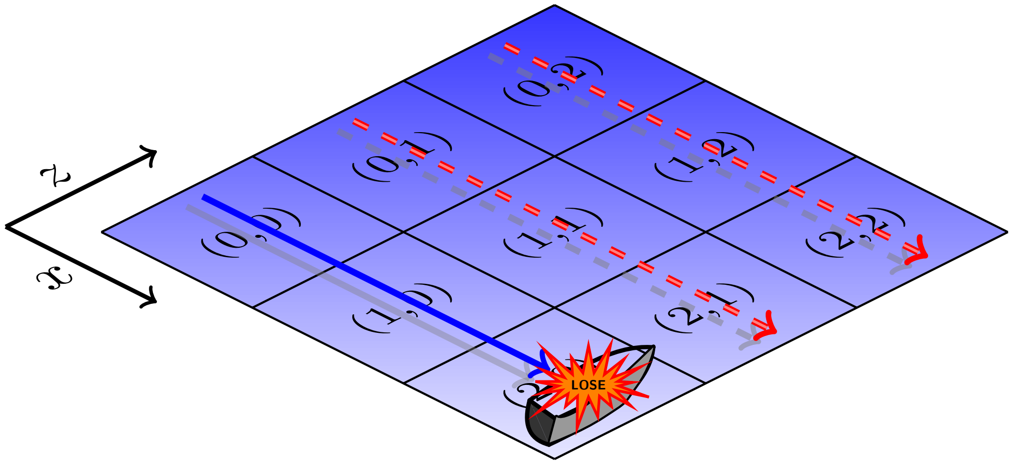

Figure 2: The Torpedo Game is a pacificist alternative to *Battleship* where the aim is to avoid sinking Alice's ship, here depicted in dimension 3.

For $d = 2$, the Torpedo Game is equivalent to a $(2,1)_2$ (Q)RAC with an additional question: along with the possibility of being asked to retrieve of one the individual input dits, Bob could be asked to retrieve relative information in the form of their parity $x \oplus z$.[2]

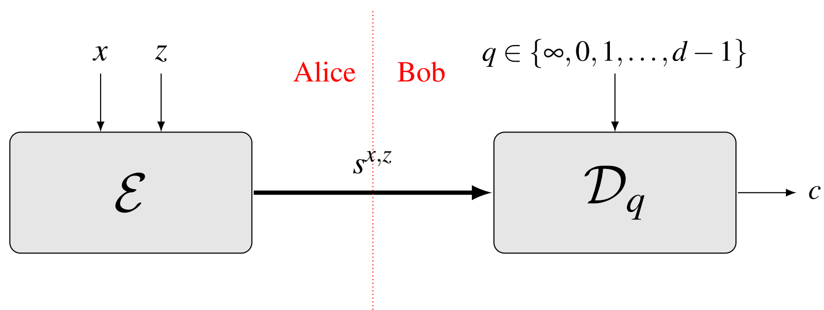

Figure 3: Operational description of the Torpedo Game: Alice receives dits x and z and sends a single message (qu)dit $s^{x,z}$ via the encoding $\mathcal{E}$. Bob is asked a question $q \in \{\infty, 0, \ldots, d-1\}$, performs decoding $\mathcal{D}_q$, and outputs c which should satisfy the winning conditions given by $w_q(x,z)$ with high probability.

2.1 Why the *Torpedo Game*?

The above game may be framed as cooperative, pacifist alternative to the popular game *Battleship*, in which Alice and Bob, finding themselves on opposing sides in a context of naval warfare, wish to subvert the conflict and cooperate to avoid casualities while not directly disobeying orders.

We take the input dits received by Alice as designating the coordinates in which she is ordered by her commander to position her one-cell ship on the affine plane of order d. We may think of the affine plane as a toric $d \times d$ grid, with x designating the row and z the column. E.g. in Fig. 4 we identify the top edge with the bottom edge and the left edge with the right edge.

[2] To see this note e.g. that $\neg x = \{-x\}$ since arithmetic is modulo 2, and returning $-x$ is equivalent to returning x since the parties know that outputs are in $\mathbb{Z}_d$, etc.

Bob is a naval officer on the opposing side who is ordered by his commander to shoot a torpedo along a line of the grid with slope specified by $q \in Q$. The ∞ question requires Bob to shoot along some row, and the 0 question requires Bob to shoot along some column, etc. However, Bob retains the freedom to choose which row, or column, or diagonal of given slope, as the case may be. More precisely, upon receiving q Bob must shoot along one of the lines $qx - z = c$ when $q \neq \infty$, and $x = c$ when $q = \infty$. However, he is free to choose the constant dit c.

Alice and Bob wish to coordinate to avoid casualities, while still obeying their explicit orders. To do so Alice may communicate a single (qu)dit to Bob – greater communication may risk revealing her position should it be intercepted. Based on this Bob must choose his c in such a way that he avoids Alice's ship.

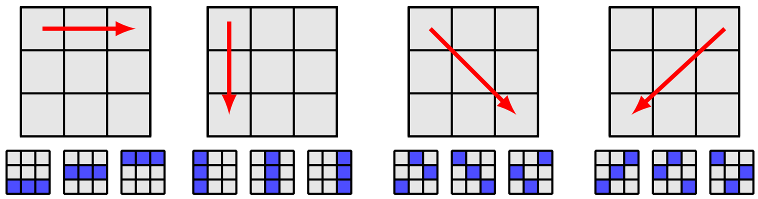

Figure 4: The red arrows depict the directions or slopes ($\infty, 0, 1, 2$, respectively) along which Bob may be asked to shoot in the $d = 3$ Torpedo Game. For each direction, Bob has three possibilities, depicted by the blue lines. On the affine plane of order 3, each group of three blue cells form a line.

2.2 Optimal Classical Strategies

Without loss of generality, consideration will be restricted to deterministic strategies, probabilistic strategies always being expressible as convex mixtures of these, which by convexity cannot improve the success probability. A deterministic strategy consists of an encoding function $\mathcal{E} : \mathbb{Z}_d^2 \to \mathbb{Z}_d$, along with $d+1$ decoding functions $\mathcal{D}_q : \mathbb{Z}_d \to \mathbb{Z}_d$, one for each $q \in Q$. In relation to Fig. 3 the message to be communicated by Alice upon receiving input (x,z) is the dit $s^{x,z} := \mathcal{E}(x,z)$, and the output to be returned by Bob upon receiving message s and question q is the dit $c := \mathcal{D}_q(s)$.

It is useful to adopt a geometric perspective (see, e.g., Fig. 8) whereby the encoding function partitions the affine plane of order d into at most d equivalence classes. Given input coordinates (x,z) Alice communicates a label for the equivalence class that they belong to and Bob makes his choice of output as a function of this information, using decoding functions that minimise the average probability of intersecting with cells of the given equivalence class.

More precisely, let $[\![(x,z)]\!]_{\mathcal{E}} := \mathcal{E}^{-1} \circ \mathcal{E}(x,z)$ be the equivalence class to which (x,z) belongs under encoding function $\mathcal{E}$. For all $c \in \mathbb{Z}_d$ and $q \in \{\infty, 0, \ldots, d-1\}$ let $L_q^c := \{(x,z) \mid qx - z = c\}$ be the set of points in the line $qx - z = c$. Decoding functions are chosen to minimise $|L_q^c \cap [\![(x,z)]\!]_{\mathcal{E}}|$ averaged over $x, z \in \mathbb{Z}_d^2$ and $q \in \{\infty, 0, \ldots, d-1\}$. This permits expression of the classical value of the Torpedo Game as

$$\theta^C_{(2,1)_d} = \max_{\mathcal{E}} \left(\frac{1}{d^2} \frac{1}{d+1} \sum_{(x,z)\in\mathbb{Z}_d^2} \sum_{q\in Q} |[\![(x,z)]\!]_{\mathcal{E}}| \left(1 - \frac{\min_{c\in\mathbb{Z}_d} \left(|L_q^c \cap [\![(x,z)]\!]_{\mathcal{E}}|\right)}{|[\![(x,z)]\!]_{\mathcal{E}}|} \right) \right). \tag{3}$$

Optimal Strategies for $d = 2$ and $d = 3$ In general there are d^{d^2} partitions of a $d \times d$ grid. For low dimensions the expression in Eq. (3) can be evaluated by exhaustive search over partitions. For dimension

2 and 3 we find

$$\theta^C_{d=2} = \frac{3}{4} \quad \text{and} \quad \theta^C_{d=3} = \frac{11}{12}. \tag{4}$$

Strategies that attains these values are depicted in Sec. A in Fig. 7 and in Fig. 8.

Optimal Strategies for $d = 5$ and beyond. As d increases it quickly becomes infeasible to perform an exhaustive search over all partitions. We have, however, found perfect classical strategies, i.e. strategies that win with probability 1, for $d = 5$ (see Sec. A, Fig. 9) up to $d = 23$. This leads us to conjecture that there exists a perfect classical strategy for any $d > 5$,

$$\textbf{Conjecture:} \quad \theta^C_{d\geq 5} = 1. \tag{5}$$

2.3 Sequential Version

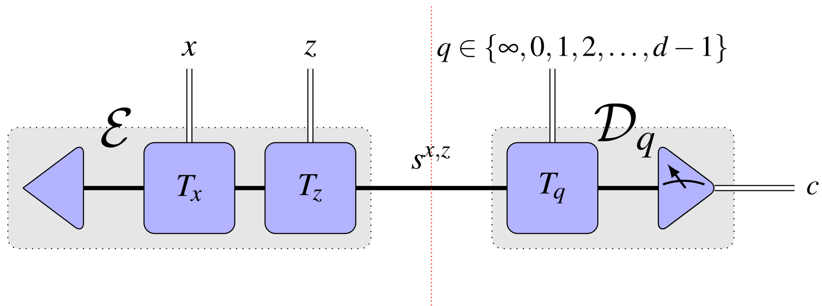

Figure 5: Sequential version of the Torpedo Game with fixed preparation and fixed measurement.

At this point we note that in operational terms strategies for the Torpedo game (Fig. 3) can equivalently be expressed in a transformation-based form (Fig. 5). In fact this equivalence holds more generally for any $(2,1)_d$ information retrieval task. This makes connections with other transformation-based protocols considered in [20, 16, 32, 27], and facilitates our contextuality analysis in Sec. 4. We refer to the operational description in Fig. 3 as the (Q)RAC version and the description in Fig. 5 as the sequential version of the Torpedo Game. The equivalence holds for both quantum and classical strategies.

Proposition 1. *Classical and quantum strategies for any $(2,1)_d$ information retrieval task can be equivalently expressed in RAC or sequential operational form.*

3 The Discrete Wigner Function

It is possible to represent finite-dimensional quantum states as quasi-probability distributions over a phase space of discrete points. Wootters [24, 42] introduced a method of constructing discrete Wigner functions (DWF) based on finite fields, wherein vectors from a complete set of mutually unbiased bases in $\mathbb{C}^d$ are put in one-to-one correspondence with the lines of a finite affine plane of order d. This geometric picture of the DWF is useful for visualizing our Torpedo Game as exemplified in Fig. 4, where each distinct orthonormal basis corresponds to a set of d parallel (non-intersecting) lines.

Gross [25] singled out one particularly symmetric definition of DWF that obeys the discrete version of Hudson's Theorem. This theorem says that an odd-dimensional pure state is non-negatively represented in the DWF if and only if it is a stabilizer state (defined blow). The discrete Hudson's Theorem has remarkable implications, providing large classes of quantum circuit with a local hidden variable model that enables efficient simulation [39, 33] . Clearly, negativity in this DWF is a necessary prerequisite for quantum speed-up. Howard *et al.* [30] showed that this negativity actually corresponds to contextuality with respect to Pauli measurements, thereby establishing the operational utility of contextuality for the gate-based model of quantum computation (particularly in a fault-tolerant setting). The equivalence of Wigner negativity and contextuality was established by deriving a noncontextuality inequality using the graph-theoretic technique of Cabello, Severini and Winter [12] which extends Kochen-Specker type state-independent proofs to the state-dependent realm. This proof (and a subsequent alternate proof [19]) requires that, as well as the system displaying Wigner negativity, a second ancillary system must be present in order to have a sufficiently rich set of available measurements.

3.1 Formalism

The discrete Wigner function is both foundationally interesting as well as practically relevant for fault-tolerant quantum computing via its link with so-called "stabilizer states". The qudit versions of the X and Z Pauli operators are

$$X|k\rangle = |k+1\rangle$$
$$Z|k\rangle = \omega^k |k\rangle$$

where $\omega = \exp(2\pi i/d)$ and arithmetic is modulo d. The qudit Pauli group has elements which are products of (powers of) these operators e.g. $X^x Z^z$ for $x,z \in \mathbb{Z}_d$. A unitary U stabilizes a state $|\psi\rangle$ if $U|\psi\rangle = |\psi\rangle$. A stabilizer state is the unique n-qudit state stabilized by a subgroup of size d^n of the Pauli group. Equivalently, stabilizer states may be understood as the image of computational basis states under the Clifford group, which is the set of unitaries that map the Pauli group to itself under conjugation.

For an arbitrary $d \times d$ Hermitian operator Q of unit trace (typically a density matrix), its Wigner representation will consist of d^2 real quasi-probabilities $W_{x,z}$ for $x,z \in \mathbb{Z}_d$. In particular, the quasi-probability associated with the point $(x,z) \in \mathbb{Z}_d^2$ is given by

$$W_{x,z} = \frac{1}{d}\mathrm{Tr}(QA_{x,z})$$

where $A_{x,z}$ are the so-called phase point operators to be defined shortly. The unit trace of Q will ensure that $\sum_{x,z} W_{x,z} = 1$. Taking the magnitude $|W_{x,z}|$ of each quasi-probability will lead to $\sum_{x,z}|W_{x,z}| = 1$ if and only if the quasi-probability distribution is actually a legitimate (non-negative) discrete probability distribution. In contrast, the presence of negative quasi-probabilities entails $\sum_{x,z}|W_{x,z}| > 1$, and in fact the departure of $\sum_{x,z}|W_{x,z}|$ from unity is a sensible measure of "how negative" or "how non-classical" the DWF of an operator is [39, 40].

When working with the DWF, it is convenient to use the Weyl-Heisenberg notation and phase convention for the qudit Pauli operators i.e.

$$D_{x,z} = \omega^{2^{-1}xz} \sum_k \omega^{kz} |k+x\rangle\langle k| = \omega^{\frac{xz}{2}} X^x Z^z, \tag{6}$$

where they go by name displacement operators. The phase point operator at the origin of phase space

$A_{0,0}$ is given by the simple expression

$$A_{0,0} = \sum_{j\in\mathbb{Z}_d} |-j\rangle\langle j|\,, \tag{7}$$

and the remainder are found by conjugation with displacement operators

$$A_{x,z} = D_{x,z}A_{0,0}D^{\dagger}_{x,z}\,. \tag{8}$$

3.2 DWF and QRACs

The eigenvectors of phase point operators are objects of interest. The maximizing eigenvectors of the phase point operators in Eq. (8) (and additional ones from different choices of DWF) were used in Casaccino *et al.* [13] as the encoded messages of a $(d+1,1)_d$ QRAC. This is natural given the use of MUBs in constructing DWFs, and prominence of MUBs in the QRAC literature. If Alice receives input $\boldsymbol{k} = (k_1, k_2, \ldots, k_{d+1}) \in \mathbb{Z}_d^{d+1}$ that she encodes in $\rho_{\boldsymbol{k}}$ and transmits to Bob, then the average probability of success for the Casaccino *et al.* QRAC is

$$\frac{1}{(d+1)d^{d+1}} \sum_{\boldsymbol{k}\in\mathbb{Z}_d^{d+1}} \mathrm{Tr}\left[\rho_{\boldsymbol{k}}(\Pi_1^{k_1} + \Pi_2^{k_2} + \ldots + \Pi_{d+1}^{k_{d+1}})\right] \tag{9}$$

where Π_q^i is the projector corresponding to dit value i in Bob's q-th measurement setting. Since phase point operators are constructed using sums of projectors from MUBs i.e., $\Pi_1^{k_1} + \Pi_2^{k_2} + \ldots + \Pi_{d+1}^{k_{d+1}}$, the use of a maximizing eigenvector for $\rho_{\boldsymbol{k}}$ is natural.

In this work we instead make use of the *minimizing* eigenvectors of phase point operators. The rationale for this is two-fold (i) these eigenvectors display remarkable geometric properties with respect to the measurements in (their constituent) mutually unbiased bases, and (ii) negativity (of a state in the DWF) is the hallmark of non-classicality which has already been identified with contextuality (with the already mentioned caveat that an additional "spectator" subsystem was required). These will be seen to lead to a perfect quantum strategy for the Torpedo Game.

As previously noted in [25, 17], the eigenvectors of phase point operators Eq. (8) are degenerate—a $+1$ eigenspace of dimension $\frac{d+1}{2}$ and a -1 eigenspace of dimension $\frac{d-1}{2}$. Any state in the -1 eigenspace has an outcome that is forbidden [17, 10] in each of a complete set of MUBs. For example, let $|\psi_{0,0}\rangle = (|1\rangle - |d-1\rangle)/\sqrt{2}$ satisfying $A_{0,0}|\psi_{0,0}\rangle = -|\psi_{0,0}\rangle$. This state obeys $\mathrm{Tr}(\Pi_q^0|\psi_{0,0}\rangle\langle\psi_{0,0}|) = 0$, where Π_q^0 is the 0-th eigenvector in the q-th basis. More specifically, Π_q^0 is the projector corresponding to the $\omega^0 = +1$ eigenvector of displacement operator $\{D_{0,1}, D_{1,0}, D_{1,1}, \ldots, D_{1,d-1}\}$, corresponding to mutually unbiased measurement bases $q \in \{\infty, 0, 1, \ldots, d-1\}$ respectively. The related states $|\psi_{x,z}\rangle = D_{x,z}|\psi_{0,0}\rangle$, which are eigenstates $A_{x,z}|\psi_{x,z}\rangle = -|\psi_{x,z}\rangle$, obey

$$\mathrm{Tr}\left[|\psi_{x,z}\rangle\langle\psi_{x,z}|\,(\Pi_\infty^x + \Pi_0^{-z} + \Pi_1^{x-z} + \cdots + \Pi_{d-1}^{(d-1)x-z})\right] = 0\,, \tag{10}$$

which implies that probability of the relevant outcome (outcome x in the first basis, $-z$ in the second basis, etc.) in each of the MUBs is zero: cf. Equation 2. The general expression for odd power-of-prime d is proven in [29, 7].

3.3 Quantum Perfect Strategy for the Torpedo Game

From Eq. (10) it follows that there is a perfect quantum strategy for the dimension d Torpedo game for any for odd power-of-prime d:

1. Upon receiving dits x and z Alice sends the following state to Bob:

$$|\psi_{x,z}\rangle = D_{x,z}|\psi_{0,0}\rangle = D_{x,z}\left(\sqrt{2}^{-1}(|1\rangle - |-1\rangle)\right). \tag{11}$$

2. Bob receives $|\psi_{x,z}\rangle$ and is asked a question $q \in \{\infty, 0, \ldots, d-1\}$. He measures the state in the MUB corresponding to q and outputs the dit corresponding to the measurement outcome.

This quantum strategy wins the Torpedo Game deterministically, i.e. with probability 1.

An analogous strategy can be employed for the dimension 2 Torpedo Game, using message states $|\psi_{x,z}\rangle = X^x Z^z |\psi_{0,0}\rangle$ where $|\psi_{0,0}\rangle\langle\psi_{0,0}| = \frac{1}{2}\left(\mathbb{I} + (X+Y+Z)/\sqrt{3}\right)$ and X, Y and Z are the usual qubit Pauli spin matrices. In the $d=2$ case it does not constitute a perfect strategy, but still achieves an advantage over classical strategies. It is not known at this stage whether this is an optimal quantum strategy, but we can leverage the fact that the $(3,1)_2$ (Q)RAC attributed to Isaac Chuang is at least as hard to win as the Torpedo Game. We obtain

$$\theta^{Q}_{d=2} \gtrsim 0.79 \quad \text{and} \quad \theta^{Q}_{d\geq 3} = 1\,. \tag{12}$$

Comparing these with the classical bounds (see Sec. 2.2) we obtain the ratios

$$\frac{\theta^{Q}_{d=2}}{\theta^{C}_{d=2}} \gtrsim 1.053 \quad \text{and} \quad \frac{\theta^{Q}_{d=3}}{\theta^{C}_{d=3}} \simeq 1.091\,. \tag{13}$$

By comparison, it was shown in [38] that the classical and quantum values of the $(4,1)_3$ (Q)RAC are $\frac{16}{27}$ and 0.637, respectively, giving a ratio of $\frac{\theta_q}{\theta_c} \simeq 1.075$. Thus the $d=3$ Torpedo Game is an information retrieval task that admits a greater quantum-over-classical advantage than the standard (Q)RAC for a comparable communication scenario.

3.4 The Sequential Version

Another way to verify Eq. (10), useful for the sequential version of the Torpedo Game, uses the matrix elements of $A_{x,z}$ combined with the Clifford gates that map the computational basis to each of the additional measurement bases. For this we use the symplectic representation of the Clifford group (the expressions below hold for odd prime d, but in the odd prime power case $d = p^n$ one should replace $\mathbb{Z}_d$ with $\mathbb{F}_d$). Clifford group elements are written as $C = D_{x,z}U_F$ [6] where

$$F = \begin{pmatrix} \alpha & \beta \\ \gamma & \varepsilon \end{pmatrix}$$

is an element of the symplectic group $\mathrm{SL}(2,\mathbb{Z}_d)$ (entries of F are in $\mathbb{Z}_d$ and $\det F = 1 \bmod d$), and

$$U_F = \begin{cases} \frac{1}{\sqrt{d}} \sum_{j,k=0}^{d-1} \omega^{2^{-1}\beta^{-1}\left(\alpha k^2 - 2jk + \varepsilon j^2\right)} |j\rangle\langle k| & \beta \neq 0 \\ \sum_{k=0}^{d-1} \omega^{2^{-1}\alpha\gamma k^2} |\alpha k\rangle\langle k| & \beta = 0 \end{cases}$$

The matrix representation [42] of a phase point operator is

$$\left(A_{x,z}\right)_{j,k} = \delta_{2x,j+k}\,\omega^{z(j-k)} \tag{14}$$

and so $\langle k|A_{x,z}|k\rangle = \delta_{k,x}$ is the likelihood of getting outcome k in a computational basis measurement of $A_{x,z}$. The Clifford unitaries $\{U_\infty, U_0, \ldots, U_{d-1}\}$ that map $Z = D_{0,1}$ to $\{D_{0,1}, D_{1,0}, \ldots, D_{1,d-1}\}$ are

$$\{U_\infty, U_0, \ldots, U_{d-1}\} = \left\{\mathbb{I}, HS^0, \ldots, HS^{d-1}\right\} = \left\{U_{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)}, U_{\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)}, \ldots, U_{\left(\begin{smallmatrix} d-1 & -1 \\ 1 & 0 \end{smallmatrix}\right)}\right\}, \tag{15}$$

where H and S are the qudit versions of the Hadamard and Phase gate respectively. Using Eq. (14), and the fact that $U_F A_{x,z} U_F^\dagger = A_{x',z'}$ where $\begin{pmatrix} x' \\ z' \end{pmatrix} = F \begin{pmatrix} x \\ z \end{pmatrix}$ [24, 25], it is straightforward to verify that

$$\begin{aligned} \langle k| U_\infty A_{x,z} U_\infty^\dagger |k\rangle &= \delta_{k,x} \\ \langle k| U_0 A_{x,z} U_0^\dagger |k\rangle &= \delta_{k,-z} \\ &\vdots \\ \langle k| U_{d-1} A_{x,z} U_{d-1}^\dagger |k\rangle &= \delta_{k,(d-1)x-z} \end{aligned} \tag{16}$$

For odd prime-power $d > 3$, the -1 eigenspace of $A_{x,z}$ has rank $(d-1)/2$. We can abuse notation slightly by referring to the normalized projector onto this eigenspace as $|\psi_{x,z}\rangle\langle\psi_{x,z}|$. The final step is to realise that $|\psi_{x,z}\rangle\langle\psi_{x,z}| = \frac{1}{d-1}(\mathbb{I} - A_{x,z})$ so that by linearity, and in agreement with Eq. (10) earlier,

$$\begin{aligned} \mathrm{Tr}\left(|\psi_{x,z}\rangle\langle\psi_{x,z}|\Pi_\infty^k\right) &= \langle k| U_\infty \frac{1}{d-1}(\mathbb{I} - A_{x,z}) U_\infty^\dagger |k\rangle = \frac{1}{d-1}(1-\delta_{k,x}) \\ \mathrm{Tr}\left(|\psi_{x,z}\rangle\langle\psi_{x,z}|\Pi_0^k\right) &= \langle k| U_0 \frac{1}{d-1}(\mathbb{I} - A_{x,z}) U_0^\dagger |k\rangle = \frac{1}{d-1}(1-\delta_{k,-z}) \\ &\vdots \\ \mathrm{Tr}\left(|\psi_{x,z}\rangle\langle\psi_{x,z}|\Pi_{d-1}^k\right) &= \langle k| U_{d-1} \frac{1}{d-1}(\mathbb{I} - A_{x,z}) U_{d-1}^\dagger |k\rangle = \frac{1}{d-1}(1-\delta_{k,(d-1)x-z}). \end{aligned} \tag{17}$$

Any state in the -1 eigenspace of $A_{x,z}$ wins the Torpedo Game with unit probability, but for concreteness we choose the state Eq. (11).

Sequential Version of the Optimal Quantum Strategy As observed in Sec. 2.3, any quantum strategy for the RAC version of the Torpedo Game admits an equivalent strategy for the sequential version. An optimal quantum strategy in sequential operational form takes as fixed preparation $|\psi_{0,0}\rangle$ and as fixed measurement Z. The transformations controlled by x, z, and q are X^x, Z^z, and U_q, respectively, where the unitaries U_q are those defined in Eq. (15).

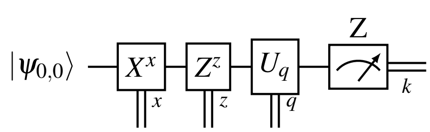

Figure 6: A perfect strategy in sequential operational form for the dimension d Torpedo Game for odd power-of-prime d.

4 Sequential Contextuality

Information retrieval tasks with quantum-over-classical advantage in communication scenarios, like (Q)RACS or the Torpedo Game, highlight a difference between the information-carrying capacities of qudits and dits. It might be remarked that such a difference is a consequence of the different geometries of the respective state spaces. In this section, we aim for a sharper analysis of the source of the advantage in terms of a feature known as sequential contextuality[3] [32].

A major theme in the foundations of quantum mechanics is to attempt to explain empirical behaviours that appear non-intuitive from a classical perspective, e.g. the EPR Paradox [21], by providing a description at a deeper level than the quantum one at which more classically intuitive properties may be restored. Such a description is typically formalised as a hidden variable, or ontological, model [36]. The celebrated no-go theorems of quantum foundations, like Bell's Theorem [8] and the Bell–Kochen–Specker Theorem [9, 31], prove however that certain *non-classical* features of the empirical behaviours of quantum systems are necessarily inherited by any such model.

Aside from their foundational importance, non-classical features of quantum systems are also increasingly investigated for their practical utility. For instance, in previous work involving present authors, it has been shown that Bell–Kochen–Specker contextuality is a prerequisite for quantum speed-up [30] and quantifies quantum-over-classical advantage in a variety of informational tasks [1]. It is this perspective that is chiefly of interest here. To this end, following the approach instigated in [32], we focus not on features that must be present in *all* possible ontologies, but on features that are necessarily present in appropriate computational (or in this case communicational) ontologies only: in particular, for analysing qudit-over-dit advantage we will be concerned with dit ontologies.

4.1 Empirical and ontological models

Any strategy, classical, quantum, or otherwise, for an information retrieval task in an $(n,m)_d$ communication scenario will give rise to an empirical behaviour. This may be described formally as an empirical model; that is a set $e = \{e_{i,q}\}$ of probability distributions over the output set $\mathbb{Z}_d$, one for each combination of input string $i \in \mathbb{Z}_d^n$ and question $q \in Q$. A combination of an input string and a question will be referred to here as a *context*. Empirical models were introduced for measurement scenarios in [2], and employed for sequential scenarios in [32].

For ontological models with dit ontology we posit a space of ontic states $\mathbb{Z}_d$. Preparation of a (quantum) state s at the operational level is modelled as inducing an ontic state, sampled according to a probability distribution on $\mathbb{Z}_d$ that for convenience we represent as a real vector $\boldsymbol{\lambda}_s$ such that $\boldsymbol{\lambda}_s \geq 0$ and $|\boldsymbol{\lambda}_s| = 1$. A transformation T_i at the operational level is modelled as a left-stochastic $d \times d$ real matrix $\mathcal{T}_i$. A measurement M_q at the operational level is modelled as a left-stochastic matrix $\mathcal{T}_q$ followed by reading or sampling of the dit according to the final distribution on ontic states. An ontological model realises an empirical model e if there exists a probability distribution $\boldsymbol{\lambda}$ such that for all $i \in \mathbb{Z}_d^n$ and $q \in Q$,

$$e_{i,q} = \mathcal{T}_q \mathcal{T}_i \boldsymbol{\lambda} \,. \tag{18}$$

4.2 Sequential (non)contextuality

An ontological model is sequential noncontextual if it preserves sequential composition of transformations. More precisely, sequential noncontextuality requires that for any finite sequence of transformations $T_{\mathrm{seq}} = T_j \circ T_{j-1} \circ \cdots \circ T_1$ at the operational level, it holds on the ontological level that

$$\mathcal{T}_{\mathrm{seq}} = \mathcal{T}_j \circ \mathcal{T}_{j-1} \circ \cdots \circ \mathcal{T}_1 \,, \tag{19}$$

and additionally requires that for any transformaton T_k its ontological representation $\mathcal{T}_k$ is context-independent, in the sense that it does not change depending on which sequence T_k is performed in. An empirical model is said to be sequentially contextual (with respect to a dit ontology) if it cannot be realised by any sequentially noncontextual dit ontological model.

[3] For an earlier treatement of an advantage in similar terms see also [43].

A sequentially contextual ontological model on the other hand could either describe transformations differently depending on which context they were used in, or violate Eq. 19. In both cases this form of non-classicality, just like other forms of non-classicality, would indicate that the whole ($\mathcal{T}_{\text{seq}}$) is somehow more than just the sum of its parts (the $\mathcal{T}_k$).

4.3 Quantifying Contextuality

Given an information retrieval task in an $(n,m)_d$ scenario, convex combinations of empirical models are defined by context-wise combinations of the constituent probability distributions. Empirical models for a given task are thus closed under convex combinations, inheriting this property from probability distributions.

Given any empirical model e, we can consider convex decompositions of the form

$$e = \omega e^{\text{NC}} + (1-\omega)e', \tag{20}$$

where e^{NC} and e' are empirical models for the same task, and e^{NC} is noncontextual. The maximum of ω over all such decompositions is referred to as the noncontextual fraction of e, written $\text{NCF}(e)$. Similarly, the **contextual fraction** of e is $\text{CF}(e) := 1 - \text{NCF}(e)$. This provides a measure of contextuality in the interval $[0,1]$, where $\text{CF}(e) = 0$ indicates that e is noncontextual, $\text{CF}(e) > 0$ indicates that e is contextual, and $\text{CF}(e) = 1$ indicates that e is maximally, or *strongly* contextual. By extension, if e is (strongly) contextual, we will say that a strategy giving rise to the empirical model e is (strongly) contextual. The contextual fraction was used as a measure for sequential contextuality in [32], extending a natural measure of BKS contextuality of the same name [2] which is known to have many desirable properties [1].

4.4 Contextual advantage

Proposition 2. *For $d = 2$ and $d = 3$, strong sequential contextuality with respect to dit ontology is necessary and sufficient to win the Torpedo Game deterministically.*

For sequential communication scenarios it is also possible to obtain the following more general result, of which Proposition 2 is a special case.

Theorem 3. *Given any information retrieval task expressible in a sequential communication scenario, and strategy with empirical behaviour e,*

$$\varepsilon \geq \mathsf{NCF}(e)\nu$$

where ε is the probability of failure, averaged over inputs and questions, $\mathsf{NCF}(e)$ is the noncontextual fraction of e with respect to a dit ontology with d fixed by the communication scenario, and $\nu := 1-\theta^C$ measures of the hardness of the task (where θ^C is the classical value of the task).

This provides a quantifiable relationship between quantum advantage and sequential contextuality. Inequalities of this *form* are also known to arise for a variety of other informational tasks that admit quantum advantage, with hardness measures and notions of non-classicality adapted to the particular task [1, 32, 41].

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Appendices

A Examples of Optimal Classical Strategies as Grids

- **Dimension 2**

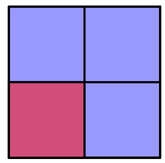

Figure 7: An optimal classical strategy for the $d = 2$ Torpedo Game. Alice uses her bit of communication to indicate in which class of the partition that she finds herself. Classes are represented here by colours.

- **Dimension 3**

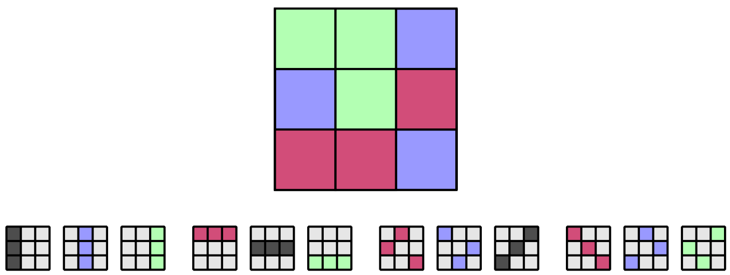

Figure 8: An optimal classical strategy for the $d = 3$ Torpedo Game. Alice uses her dit of communication to indicate in which equivalence class (represented by same coloured cells) of the large grid partition she finds herself. The smaller grids (cf. Fig. 4) show where Bob chooses to shoot, given a direction and a colour. For the first directon, when asked to shoot vertically in the grid, notice that Bob may avoid Alice with certainty if she is in either of the blue or green partitions. Lines that avoid Alice with certainty are depicted in the corresponding colour, whereas black lines intersect with Alice's position with probability $\frac{1}{3}$. Overall, this strategy wins the Torpedo Game with probability $\frac{1}{4}(\frac{8}{9}+\frac{8}{9}+\frac{8}{9}+1) = \frac{11}{12}$.

- **Dimension 5**

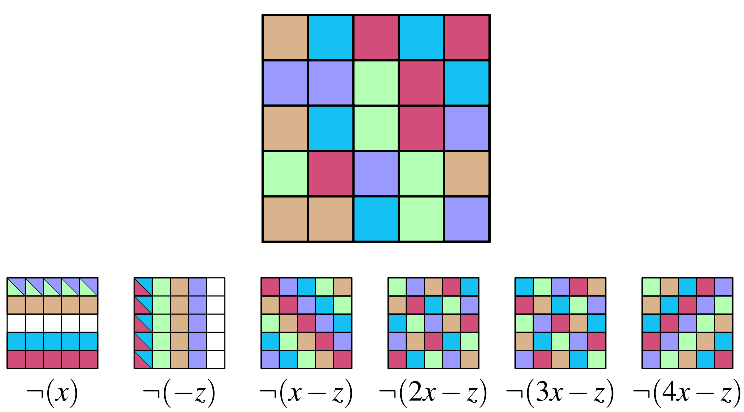

Figure 9: A perfect classical strategy for the $d = 5$ Torpedo Game. As Fig. 8, same coloured cells belong to the same partition. The lines that avoid Alice are depicted below for every questions Bob can be asked.

B Proofs

Proof of Proposition 1

Proposition. *Classical and quantum strategies for any* $(2,1)_d$ *information retrieval task can be equivalently expressed in RAC or sequential operational form.*

Proof. Since the initial preparation is fixed in the sequential version, it is clear that the encoding step can always be re-expressed as a map $\mathcal{E} : \mathbb{Z}_d \times \mathbb{Z}_d \to \mathbb{Z}_d$ (or $\mathcal{E} : \mathbb{Z}_d \times \mathbb{Z}_d \to \mathbb{C}^d$) as in the (Q)RAC version. Conversely, a strategy for the RAC version can be expressed as a strategy for the sequential version by setting T_x to always output x and taking for T_z the encoding map $\mathcal{E}$ from the RAC version, while for the QRAC version T_x outputs $|x\rangle$, and T_z is simply a Z measurement subsequently composed with the encoding map $\mathcal{E}$.

Similarly, for the decoding step it is clear that the sequential version can always be expressed as a map in the form of the (Q)RAC version. In the classical case, for the converse it suffices to take for T_q the decoding map $\mathcal{D}_q$ from the RAC version, and as fixed measurement the identity map on $\mathbb{Z}_d$. In the quantum case, the converse follows from he observation that any projection-valued measurement can be expressed as a unitary transformation followed by a fixed measurement in the Z basis. □

Proof of Proposition 2

Proposition. *For* $d = 2$ *and* $d = 3$*, strong sequential contextuality with respect to dit ontology is necessary and sufficient to win the Torpedo Game deterministically.*

Proof. Let $\mathbf{f}_k$ denote the kth basis vector in the vector space $\mathbb{Z}_d^2$ over $\mathbb{R}$. For all $x, z \in \mathbb{Z}_d$, $q \in Q$, let P_x^1, $\mathcal{T}_z^2$ and $\mathcal{T}_q^3$ be the ontological matrices representing the transformations E_x, E_z and U_q respectively. Contexts are labelled by the input-question combinations (x, z, q).

Suppose an ontological model realises an empirical model that wins the Torpedo Game deterministically. Then by Eq. (18) and Eq. (2) it must hold for all $x, z \in \mathbb{Z}_d$, $q \in Q$ that

$$\mathcal{T}_q^3 \mathcal{T}_z^2 \mathcal{T}_x^1 \, \boldsymbol{\lambda} \cdot \mathbf{f}_{v(x,z,q)} = 0 \tag{21}$$

where $\boldsymbol{\lambda}$ is the initial probability distribution over ontic states, and $v(x,z,q)$ is a function specifying the losing condition given input (x,z) and question q,

$$v(x,z,q) := \begin{cases} x & q = \infty \\ -z & q = 0 \\ x - z & q = 1 \\ 2x - z & q = 2 \end{cases}$$

A sequentially noncontextual realisation would require the system of linear equations Eq. (21) to be *jointly* satisfiable. We know that this cannot be possible since it would provide a RAC perfect strategy for the $d = 2$ and $d = 3$ Torpedo Games deterministically, violating the optimal bounds given in Eq. (4). On the other hand, it is always possible to obtain a contextual realisation, by taking context-wise solutions to Eq. (21).

It can further be observed that if any fraction of an empirical model e can be described noncontextually, i.e. $\mathsf{NCF}(e) = p > 0$, then with an average probability at least p the empirical model e fails in the Torpedo Game. Therefore, to win the Torpedo Game deterministically requires strong contextuality.

□

Remark It is possible to perform a brute-force search over all possible deterministic left stochastic transformations in order to check how many of the linear equations in Eq. (21) can be jointly satisfied. As expected, for dimension 2, we find that at most 9 out of 12 equations in Eq. (21) may be jointly satisfied, matching the classical bound of Eq. (4). For $d = 3$, we were unable to perform the brute-force calculation due to the size of the search space. However the classical bound of Eq. (4) found by means of our grid partioning method implies that at most 33 out of 36 equations in Eq. (21) may be jointly satisfied.

A $d = 3$ solution that attains the classical value of $\frac{11}{12}$, i.e. that satisfies jointly 33 of the 36 equations from Eq. (21), using reversible gates only, is the following:

$$\begin{array}{llll} \mathcal{T}_{x=0} = \mathbb{I} & \mathcal{T}_{x=1} = \mathbb{I} & \mathcal{T}_{x=2} = \oplus 1 & \\ \mathcal{T}_{z=0} = \mathbb{I} & \mathcal{T}_{z=1} = \oplus 2 & \mathcal{T}_{z=2} = \oplus 1 & \\ \mathcal{T}_{j=\infty} = \mathbb{I} & \mathcal{T}_{j=0} = \oplus 1 & \mathcal{T}_{j=1} = \oplus 2 & \mathcal{T}_{j=2} = \oplus 1 \end{array} \tag{22}$$

This strategy can be implemented by states, transformations and measurements that are non-negatively represented in the discrete Wigner function, taking the stabilizer state $|0\rangle$ as initial state and representing the above permutation transformations in the obvious way. Thus the classical bound is saturated by a non-negative quantum strategy.

Proof of Theorem 3

Theorem. *Given any information retrieval task expressible in a sequential communication scenario, and strategy with empirical behaviour e,*

$$\varepsilon \geq \mathsf{NCF}(e)\nu$$

where ε is the probability of failure, averaged over inputs and questions, $\mathsf{NCF}(e)$ is the noncontextual fraction of e with respect to a dit ontology with d fixed by the communication scenario, and $\nu := 1 - \theta^C$ measures of the hardness of the task (where θ^C is the classical value of the task).

Proof. We can decompose the resource empirical model as:

$$e = \mathsf{NCF}(e)e^{\mathrm{NC}} + \mathsf{CF}(e)e'$$

where e' is necessarily strongly contextual. From this convex decomposition, we obtain that the probability of success using the empirical model e reads:

$$p_{S,e} = \mathsf{NCF}(e)p_{S,e^{\mathrm{NC}}} + \mathsf{CF}(e)p_{S,e'}$$

where $p_{S,e^{\mathrm{NC}}}$ and $p_{S,e'}$ are the average probabilities associated with empirical models e^{NC} and e' respectively. At best, e' wins with probability 1 and thus:

$$p_{S,e} \leq \mathsf{NCF}(e)p_{S,e^{\mathrm{NC}}} + \mathsf{CF}(e)$$
$$\varepsilon \geq \mathsf{NCF}(e)\varepsilon_{e^{\mathrm{NC}}}$$

where $\varepsilon_{e^{\mathrm{NC}}} = 1 - p_{S,e^{\mathrm{NC}}}$ is the average probability of failure associated with e^{NC}. Since the latter is noncontextual, we know that the minimum probability of failure is $\nu_d = 1 - \theta^C$, where θ^C is the classical value of the game. Then $\varepsilon_{e^{\mathrm{NC}}} \leq \nu_d$, from which we obtain the desired inequality:

$$\varepsilon \geq \mathsf{NCF}(e)\nu_d$$

□

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