Contextual advantage for noisy one-shot classical communication assisted by entanglement

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We consider the problem of one-shot classical communication assisted by entanglement. In the zero-error regime, a strategy to increase the one-shot zero-error capacity of a family of classical channels based on the Kochen-Specker (KS) theorem was proposed by Cubitt *et al.*, Phys. Rev. Lett. 104, 230503 (2010). In the generic regime of noisy entangled states and/or noisy nonprojective local measurements, the noiseless KS argument doesn't apply and it's not possible to achieve an enhancement of the one-shot zero-error capacity using the scheme of Cubitt *et al.* We therefore study the enhancement of the *one-shot success probability* of sending a fixed number of classical messages rather than the one-shot zero-error capacity of the classical channel. We show the necessity *and* sufficiency of contextuality for an advantage over the classical one-shot success probability. Our analysis is general in that it also applies to classical channels which do not admit the possibility of one-shot zero-error communication and/or its enhancement based on the KS theorem, e.g., the classical channel of Prevedel *et al.*, Phys. Rev. Lett. 106, 110505 (2011), where only the one-shot success probability can be enhanced. We thus identify contextuality as the key notion of nonclassicality responsible for enhancing the one-shot success probability of classical channels in the presence of entanglement. Under some further assumptions on the classical channel and the inference strategy (satisfied, in particular, by the Cubitt *et al.* example), we highlight the graph-theoretic properties of the channel. We show that contextuality witnessed by a preparation-measurement correlation function, following the framework of R. Kunjwal, Quantum 4, 219 (2020), is sufficient for enhancing the one-shot success probability. The hypergraph invariant – weighted max-predictability – introduced in R. Kunjwal, Quantum 3, 184 (2019), thus finds an application in certifying the usefulness of any choice of entangled state and local measurements in enabling a quantum advantage in this task.

The central promise of recent advances in quantum technologies is a quantum advantage over what's possible classically. Several protocols and systems have demonstrated this advantage, but identifying the source of this improvement is not trivial. While several nonclassical resources within quantum theory (e.g. entanglement, incompatibility, etc.) that enable this enhancement are well-studied, one would ideally like features that can be tested operationally (i.e., directly from the data) under minimal assumptions about the structure of the theory. These features would not only characterize and distinguish quantum theory from classical theories but would also provide insights about the source of quantum advantage in as theory-independent a manner as possible. In this contribution, we demonstrate that the contextuality (i.e., violation of the assumption of noncontextuality) does exactly this in the context of entanglement-assisted one-shot classical communication.

The framework for noncontextuality that we use, developed by Spekkens [6], posits that operational indistinguishablity implies ontological indistinguishability. This criterion for ontological models constrains the statistics that can be simulated by them and quantum theory yields statistics that clearly violates this restriction. We show that in entanglement-assisted classical communication, this violation yields a quantum advantage.

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The contextuality of quantum theory was first made manifest by the Kochen-Specker (KS) theorem [2]. While the original theorem only considered ideal projective measurements and used an uncolourability argument, recent work by Kunjwal and Spekkens [4, 3] operationalizes contextuality and provides noise-robust contextuality witnesses using a hypergraph invariant. They show that in the noncontextual regime, the average correlation between preparation and measurement pairs (with some equivalence-structure hypergraphs) is upper bounded by a hypergraph invariant called the *weighted max-predictability*, denoted β: $\mathrm{Corr}_{NC} \leq \beta < 1$. A violation of this inequality indicates the absence of a noncontextual ontological model, thus strengthening the case for contextuality as a central nonclassical feature of quantum theory. The regime of the KS theorem is approached when the measurements and preparations become increasingly projective/pure and $\mathrm{Corr} \to 1$.

The theme of graph-theoretic frameworks provides a natural lead-up to one-shot classical communication. The problem of zero-error one-shot messaging was reduced by Shannon to a graph coloring problem [5] [1]. The maximum number of messages that can be sent over a classical channel with zero error is the independence number of the channel's hypergraph, denoted by α. In a channel hypergraph, the input symbols are represented by vertices and hyperedges are the confusability sets associated with the output symbols. Noticing that a graph-theoretic invariant (α) bounds the zero-error capacity, *and* that in quantum theory this bound can be exceeded, one is motivated to ask if one-shot communication can be enhanced with the use of quantum resources. Cubitt et al. [1] explore this connection and provide a communication protocol that invokes Kochen-Specker sets. They demonstrate that the capacity of a classical channel with a hypergraph that matches a KS hypergraph can be increased using a maximally entangled state shared by both parties and projective measurements that constitute Kochen-Specker sets. While this is a fundamental proof-of-concept, enhancing the zero-error capacity in this manner is not experimentally feasible due to inevitable noise in any quantum implementation. We therefore generalize this protocol to allow for noisy shared entangled states and local measurements. Instead of the one-shot zero-error capacity, we use the communication success probability for a fixed number of messages as our figure of merit. With this, our notion of quantum advantage gets formalized as exceeding the maximal classical success probability. In our work, contextuality emerges as a nonclassical feature that drives this advantage. A preliminary draft is available at this link

We first provide a general framework for entanglement-assisted one-shot classical communication. We demonstrate that preparation noncontextuality is a necessary and sufficient restriction for recovering the classical shared randomness regime.[2] In the Cubitt et al. protocol, we assume that Bob (the receiver) is oblivious to the exact channel probabilities, but he still retains knowledge of the confusability sets, i.e. the hypergraph structure of the channel. We call this Bob's *context-independent guessing* (CIG). This assumption motivates operational equivalences between Bob's measurement procedures, thus paving the way for a connection with contextuality. We demonstrate necessity and sufficiency of contextuality for an advantage in the Cubitt et al. protocol. We go further and use the aforementioned hypergraph invariant to lower and upper bound the success probability of communication. We show that,

$$\eta^{min} + \mathrm{Corr}(1-\eta^{min}) \leq S \leq \eta^{max} + \mathrm{Corr}(1-\eta^{max}) \tag{1}$$

where η^{min} and η^{max} are the minimum and maximum confusability of two input symbols from Bob's perspective and depend *only* on the classical channel. Corr here witnesses contextuality on Bob's side

[1] Zero-error one-shot communication studies message encodings the enable perfect communication given a single use of the channel.

[2] Note that the necessary condition for enhancement is steerability which is more restrictive than entanglement.

by considering the correlation between the states steered by Alice onto Bob's side and Bob's local measurements. When we have projective measurements and maximally entangled states, $\mathrm{Corr} = 1$, and thus $S = 1$ irrespective of the channel probabilities, recovering the zero-error result of Cubitt *et al*. In the special case where $\eta^{min} = \eta^{max} \equiv \eta$, we have that $S = \eta + \mathrm{Corr}(1-\eta)$ and $\mathrm{Corr} > \beta$ implies that S exceeds the classical success probability.

References

[1] Toby S. Cubitt, Debbie Leung, William Matthews & Andreas Winter (2010): *Improving Zero-Error Classical Communication with Entanglement*. Phys. Rev. Lett. 104, p. 230503, doi:10.1103/PhysRevLett.104.230503. Available at `https://link.aps.org/doi/10.1103/PhysRevLett.104.230503`.

[2] Simon Kochen & E. P. Specker (1967): *The Problem of Hidden Variables in Quantum Mechanics*. Journal of Mathematics and Mechanics 17(1), pp. 59–87. Available at `http://www.jstor.org/stable/24902153`.

[3] Ravi Kunjwal (2020): *Hypergraph framework for irreducible noncontextuality inequalities from logical proofs of the Kochen-Specker theorem*. Quantum 4, p. 219, doi:10.22331/q-2020-01-10-219. Available at `https://doi.org/10.22331/q-2020-01-10-219`.

[4] Ravi Kunjwal & Robert W. Spekkens (2015): *From the Kochen-Specker Theorem to Noncontextuality Inequalities without Assuming Determinism*. Phys. Rev. Lett. 115, p. 110403, doi:10.1103/PhysRevLett.115.110403. Available at `https://link.aps.org/doi/10.1103/PhysRevLett.115.110403`.

[5] C. Shannon (1956): *The zero error capacity of a noisy channel*. IRE Transactions on Information Theory 2(3), pp. 8–19, doi:10.1109/TIT.1956.1056798.

[6] R. W. Spekkens (2005): *Contextuality for preparations, transformations, and unsharp measurements*. Phys. Rev. A 71, p. 052108, doi:10.1103/PhysRevA.71.052108. Available at `https://link.aps.org/doi/10.1103/PhysRevA.71.052108`.