How dynamics constrains probabilities in general probabilistic theories

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The following is an extended abstract for [11] which can be found at https://arxiv.org/abs/2002.05088.

1 Introduction

General probabilistic theories (GPTs) provide a framework for the study of operational theories beyond quantum theory. Within this framework quantum theory appears as one non-classical theory amongst many. Examples of GPTs (excepting classical and quantum theory) include Boxworld [7, 4, 17, 12, 1], quantum theory over the field of real numbers [13, 2, 18] or quaternions [8], theories based on Euclidean Jordan algebras [3], quartic quantum theory [19], d-balls [5, 15, 14], density cubes [6] and quantum systems with modified measurements [9].

The aim of this paper is to provide tools to systematically explore the space of non-classical systems. Rather than generating examples of non-classical systems we show how to give full classifications of families of non-classical systems which share a common dynamical structure (pure states and reversible dynamics) but different probabilistic structures (measurements and measurement outcome probabilities).

Using a notion of distance between probabilistic structures we introduce the property of rigidity of probabilistic structures: a probabilistic structure is rigid if it cannot be continuously deformed. All previously studied GPTs have rigid probabilistic structures. We find necessary and sufficient conditions on the dynamical structures for the existence of non-rigid probabilistic structures. We explore in detail an example of a non-rigid system, and show that by continuously deforming the probabilistic structure one can change the number of perfectly distinguishable states. We also introduce new families of non-classical systems and provide a full classification of a family of systems which includes both quantum theory and quartic quantum theory as special cases.

2 The OPF framework and rigidity of probabilistic structures

We provide a characterisation of single systems within the GPT framework which emphasises the pure states and reversible dynamics. This will allow us to consider families of systems with the

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same pure states and reversible dynamics, but different measurements. This is a generalisation of [9, 10, 16] where all systems with the same pure states and reversible dynamics as quantum theory were classified and their informational properties studied.

The *dynamical structure* of a system S_X is a set X acted on by a group G. We assume the action to be transitive and hence $X \cong G/H$ for some subgroup H of G. The probabilistic structure $\mathscr{F}_X$ is a set of functions $\mathbf{f} : X \to [0,1]$ which are closed under mixing and composition with group elements. For a given dynamical structure X, different choices of probabilistic structures $\mathscr{F}_X$ will lead to different equivalence classes of indistinguishable ensembles and hence different sets of mixed states.

An unrestricted probabilistic structure $\mathscr{F}_X$ is *rigid* if any other unrestricted probabilistic structure $\mathscr{F}'_X$ of the same linear dimension is at a finitely bounded distance.

3 Results

There are two main results in this work. The first is a theorem about the classification of probabilistic structures for a given dynamical structure.

Theorem 1 (Classification theorem)**.** *Let $\mathscr{D} = (G,H)$ be a transitive dynamical structure, and let us consider probabilistic structures $\mathscr{F}_{G/H}$ such that $\mathbb{R}[\mathscr{F}_{G/H}]$ is finite-dimensional. Every system $\mathscr{S}_{G/H} = (G,H,\mathscr{F}_{G/H})$ has an associated representation $\Gamma : G \to \mathrm{GL}(\mathbb{R}[\mathscr{F}_{G/H}]^*)$.*

i. Every probabilistic structure $\mathscr{F}_{G/H}$ (up to tomographic equivalence) has an associated representation Γ of the form:

$$\Gamma = \bigoplus_j \Gamma_j, \tag{1}$$

where each term $(\Gamma_j, V_j, \mathbb{R})$ is a real-irreducible representation with least one trivial subrepresentation when restricted to H.

ii. Conversely every representation of the form (1) *is associated to at least one probabilistic structure $\mathscr{F}_{G/H}$.*

iii. When (G,H) forms a Gelfand pair the correspondence between representations $(\Gamma, V, \mathbb{R})$ of the form (1) *and probabilistic structures (up to tomographic equivalence) $\mathscr{F}_{G/H}$ is one-to-one.*

iv. When (G,H) does not form a Gelfand pair then some representations $(\Gamma, V, \mathbb{R})$ of the form (1) *have infinitely-many tomographically inequivalent probabilistic structures $\mathscr{F}_{G/H}$ associated to them.*

We also identify which dynamical structures allow for non-rigid probabilistic structures.

Theorem 2. *Let $\mathscr{D} = (G,H)$ be a dynamical structure.*

1. If (G,H) is a Gelfand pair, then every unrestricted probabilistic structure $\mathscr{F}_{G/H}$ is rigid.

2. *If (G,H) is not a Gelfand pair, then there exist probabilistic structures $\mathscr{F}_{G/H}$ which are not rigid, which are those with associated representations Γ_G which admit H-invariant vectors related by invertible transformations which do not commute with Γ_G.*

A group/subgroup pair (G,H) form a Gelfand pair if and only if all irreducible representations of G contain at most one H-invariant subspace.

Following this we explore in depth an example of a non-rigid system as well as provide a full classification of all systems with pure states given by the manifold of all k-dimensional subspaces of the complex linear space $\mathbb{C}^d$ ($k < d$). This manifold is known as a Grassmann manifold $\mathrm{Gr}(k,\mathbb{C}^d)$. It is a generalisation of the space of pure states of quantum theory which is $\mathrm{P}\mathbb{C}^d \cong \mathrm{Gr}(1,\mathbb{C}^d)$

4 Discussion

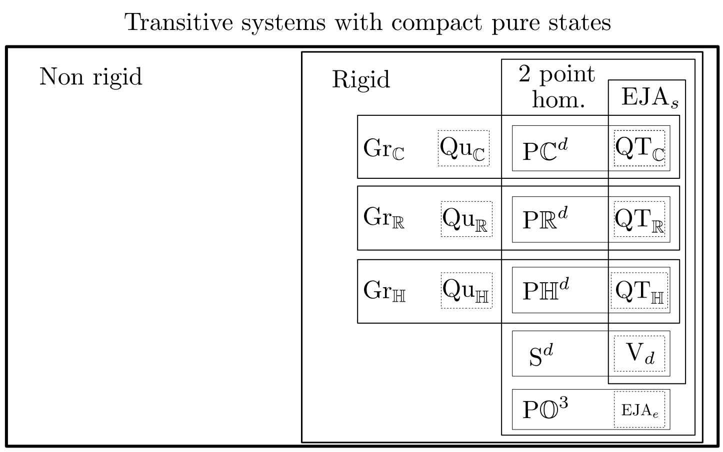

Figure 1: Map of the space of transitive non-classical systems with compact pure states $X = G/H$. '2 point hom.' stands for two point homogeneous. For $\mathbb{F}$, $\mathrm{Gr}_{\mathbb{F}}$ is the family of systems with pure states given by the Grassmann manifold $\mathrm{Gr}(\mathbb{F}^d,\mathbb{F}^k)$ for all $2 < d < \infty$, $k < d$. $\mathrm{P}\mathbb{F}^d$ is the family of systems with pure states given by projective space over $\mathbb{F}^d$ for all $1 < d < \infty$. $\mathrm{QT}_{\mathbb{F}}$ is quantum theory over $\mathbb{F}$ whilst $\mathrm{Qu}_{\mathbb{F}}$ is quartic quantum theory over $\mathbb{F}$. 'EJA_s' labels special Euclidean Jordan Algebras (EJA) and 'EJA_e' the exceptional EJA. V_d is the $d-$sphere in the standard embedding in $\mathbb{R}^{d+1}$ whilst S^d is the family of systems with pure states given by S^d (hence embeddings of S^d in $\mathbb{R}^k$ where k not necessarily equal to $d+1$). This map does not capture all the relations, namely there are 'coincidences' like the qubit being both in $\mathrm{QT}_{\mathbb{C}}$ and V_d.

By making us of representation theoretic tools for the classification of GPT systems we can further explore the space of non-classical systems. In Figure 1 we provide a summary of the new families of systems introduced, and where known GPTs figure in this classification.

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