A Diagrammatic Approach to Information Transmission in Generalised Switches

Matthew Wilson
Department of Computer Science, University of Oxford
matthew.wilson@cs.ox.ac.uk

Giulio Chiribella
Department of Computer Science, The University of Hong Kong
Department of Computer Science, University of Oxford
giulio.chiribella@cs.ox.ac.uk

We write the quantum switch a sum of diagrams in **FHILB** each of which is built from maps in the CPM-construction. The resulting picture gives alternative intuition for the activation of classical capacity of completely depolarising channels (CDPC's) and allows for generalisation to N-party switches. We demonstrate the use of these partially diagrammatic methods by deriving a permutation condition for computing the output of any N-party switch of CDPC's, we then use that condition to prove that amongst all possible terms, interferences between cyclic permutations are the uniquely maximally-normalised information transmitting terms.

1 Introduction

Quantum Shannon theory [29, 31] explores the extension of Shannon's information theory to scenarios where the information carriers are quantum systems. Recently, there has been an interest a further extension, where not only the information carriers, but also the configuration of the communication channels, can be quantum [16, 8, 27, 1, 18, 21]. These extensions allow the communication channels to be combined in more general ways than those allowed in standard quantum Shannon theory. Technically, the combination of channels is described by a quantum supermap [5], a higher-order transformation that maps channels into channels. A paradigmatic example of supermap is the quantum switch [6, 9] of two channels $\mathcal{N}^{(1)}$, $\mathcal{N}^{(2)}$ which superposes the two possible sequential compositions $\mathcal{N}^{(1)} \circ \mathcal{N}^{(2)}$ and $\mathcal{N}^{(2)} \circ \mathcal{N}^{(1)}$. The quantum switch has been shown to offer computational advantages [7, 23, 4, 17], as well as advantages in quantum metrology [22, 32].

In this paper we analyze the information processing advantages of the quantum switch from the perspective of Categorical Quantum Mechanics (CQM) [2, 13] which has been used, to give new insights into known quantum protocols [13], to design new protocols [26], and to formalise the concepts of causality [14] and causal structure [20]. Here we use the diagrammatic language of CQM to analyse information processing advantages of the quantum switch by writing each output as a sum of diagrams built from the CPM-construction [30, 12]. We focus specifically on the generalisation of the activation of capacities of completely depolarising channels (CDPC's) as in [16] to switches of general permutations of N completely depolarising channels, which we refer to as N-party switches. Previous work on this subject was done by Procopio *et al* in Ref. [24], where a formula is given for the action of the quantum switch of N partially depolarising channels $\{\mathcal{N}^{(i)}\}_{i=1}^{N}$. Here we provide an explicit expression for the output of such an N-party switch using only diagrammatic manipulations based on the algebra of permutations. This permutation condition is then used, to suggest the switch of the N cyclic permutations

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of N channels as a protocol for high capacity enhancement, the specific enhancement properties of the superposition of cyclic permutations are independently discussed in [11] and [28].

2 Preliminaries

We first review the algebraic representation of quantum channels and of the quantum switch within the Hilbert space framework of quantum mechanics. Then, we review the diagrammatic representation of quantum channels in the framework of categorical quantum mechanics.

2.1 Algebraic Presentation of a Quantum Channel

In the pure state picture of quantum mechanics, a quantum state is represented by a normalised element $|\psi\rangle$ of a Hilbert space $\mathscr{H}$, up to a global phase. Pure quantum states are then generalised to mixed states, described by trace normalised positive linear operators $\rho \in L(\mathscr{H})$ on Hilbert spaces, $L(\mathscr{H})$ denoting the set of linear operators on the Hilbert space $\mathscr{H}$.

A quantum channel $\mathcal{N} : L(\mathscr{H}) \to \mathscr{L}(\mathscr{H})$ is any transformation $\mathcal{N}(\rho)$ which is linear, trace preserving, and completely positive. For any quantum channel there exists operators $\{K_i\}$ such that $\mathcal{N}(\rho) = \sum_i K_i \rho K_i^\dagger \; \forall \rho \in L(\mathscr{H})$, which is referred to as a Kraus decomposition of a channel $\mathcal{N}$ into Kraus operators $\{K_i\}$. Two canonical examples of quantum channels are the identity channel $\mathcal{I}$ and the completely depolarising channel $\mathcal{D}$, defined as $\mathcal{I}(\rho) = \rho$ and $\mathcal{D}(\rho) = \frac{I}{d}$, respectively.

2.2 Algebraic Presentation of the Quantum Switch

Channels are transformations of states. One can also consider transformations of channels, an idea formally captured by the framework of quantum supermaps [5, 10]. The quantum switch [9] is a bipartite supermap, from a pair of channels $\mathcal{N}^{(1)}$, $\mathcal{N}^{(2)}$ and a fixed control qubit $|+\rangle\langle+|$ it produces a superposition of sequential compositions $\mathcal{N}^{(1)} \circ \mathcal{N}^{(2)}$ and $\mathcal{N}^{(2)} \circ \mathcal{N}^{(1)}$.

The quantum switch S of $\mathcal{N}^{(1)}$ and $\mathcal{N}^{(2)}$, with Kraus decompositions $\{K^{(1)}_{\mathbf{1}_i}\}_{i=1}^{d^2}$ and $\{K^{(2)}_{\mathbf{2}_i}\}_{i=1}^{d^2}$ respectively can be split into four components

$$
\begin{aligned}
S(\mathcal{N}^{(1)}, \mathcal{N}^{(2)})(\rho, |+\rangle\langle+|) = & \frac{1}{2} \sum_{\mathbf{2}_i}^{d^2} \sum_{\mathbf{1}_i}^{d^2} K^{(1)}_{\mathbf{1}_i} K^{(2)}_{\mathbf{2}_i} \rho K^{(2)^\dagger}_{\mathbf{2}_i} K^{(1)^\dagger}_{\mathbf{1}_i} \otimes |0\rangle\langle 0| \\
& + \frac{1}{2} \sum_{\mathbf{2}_i}^{d^2} \sum_{\mathbf{1}_i}^{d^2} K^{(2)}_{\mathbf{2}_i} K^{(1)}_{\mathbf{1}_i} \rho K^{(1)^\dagger}_{\mathbf{1}_i} K^{(2)^\dagger}_{\mathbf{2}_i} \otimes |1\rangle\langle 1| \\
& + \frac{1}{2} \sum_{\mathbf{2}_i}^{d^2} \sum_{\mathbf{1}_i}^{d^2} K^{(1)}_{\mathbf{1}_i} K^{(2)}_{\mathbf{2}_i} \rho K^{(1)^\dagger}_{\mathbf{1}_i} K^{(2)^\dagger}_{\mathbf{2}_i} \otimes |0\rangle\langle 1| \\
& + \frac{1}{2} \sum_{\mathbf{2}_i}^{d^2} \sum_{\mathbf{1}_i}^{d^2} K^{(2)}_{\mathbf{2}_i} K^{(1)}_{\mathbf{1}_i} \rho K^{(2)^\dagger}_{\mathbf{2}_i} K^{(1)^\dagger}_{\mathbf{1}_i} \otimes |1\rangle\langle 0| \qquad (1)
\end{aligned}
$$

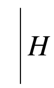
Interference between the two sequential orderings of $\mathcal{N}^{(1)}$ and $\mathcal{N}^{(2)}$ is seen in the off diagonal elements of the control qubit. When $\mathcal{N}^{(1)}$ and $\mathcal{N}^{(2)}$ are both completely depolarising channels (CDPC's), $\mathcal{N}^{(1)} = \mathcal{N}^{(2)} = \mathcal{D}$, the algebraic properties of the Kraus decomposition of $\mathcal{D}$ can be used to compute the output explicitly [16].

$$S(\mathcal{N}^{(1)},\mathcal{N}^{(2)})(\rho,|+\rangle\langle+|)=\frac{1}{2}\sum_{i,j\in\{0,1\}}\left[\delta_{ij}\frac{I}{d}+(1-\delta_{ij})\frac{\rho}{d^2}\right]\otimes|i\rangle\langle j| \tag{2}$$

The dependence of the output on ρ, implies this channel can transmit information, formally it has non-zero classical capacity [16]. The output after discarding of the control qubit is maximally mixed, information can only reach the receiver if the receiver additionally have access to the control system.

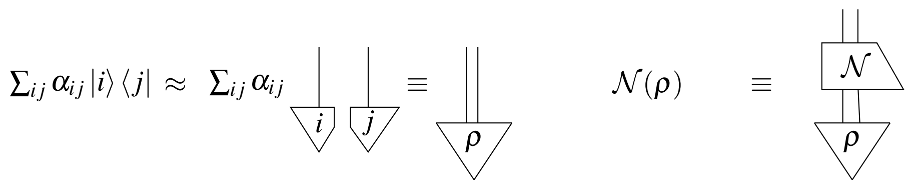
2.3 Diagrammatic Representation of a Quantum Channel

In the language of categorical quantum mechanics, Hilbert spaces are drawn as wires [13],



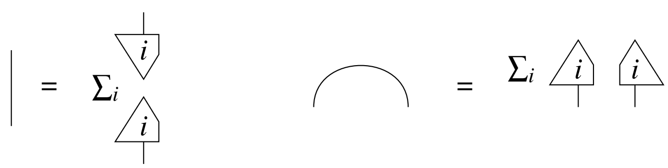
(3)

A density matrix ρ is a box with output wires (wires pointing upwards), a quantum channel $\mathcal{N}$ has input and output wires.



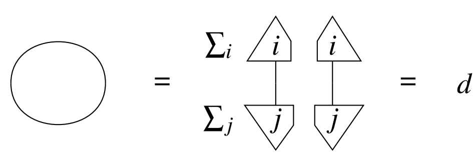
(4)

Any plain wire can be considered an identity map and expanded as a resolution of the identity, similarly for a bent wire (or “cap”) representing the trace.



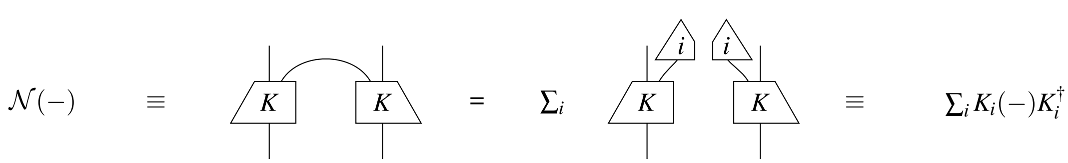
(5)

A closed loop is then the dimension of the Hilbert space $d=\dim(\mathscr{H})$,



(6)

and finally the Kraus decomposition of a map can be expressed using the bent wire.

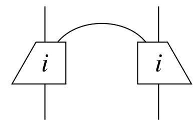


(7)

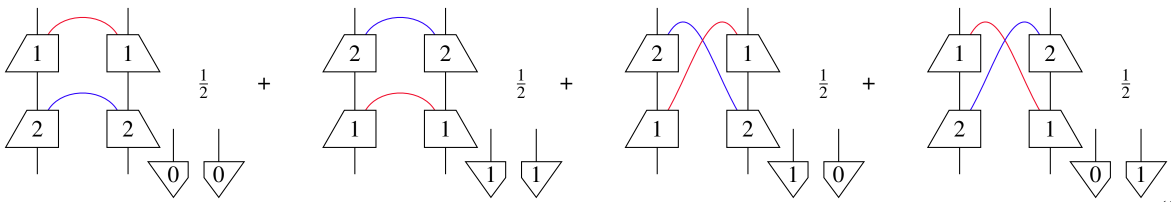
Formally the bent wire representation of a quantum channel is the CPM-construction [30]. As we will see later in this paper, the representation of the trace as a wire that gives an intuition for the wherabouts of the information flow in indefinite causal order scenarios.

3 Translating the Output of the Quantum Switch into a Sum of Diagrams

We notate each of the two quantum channels in the input of the quantum switch as $\{\mathcal{N}^{(i)}\}_{i=\{1,2\}}$ by

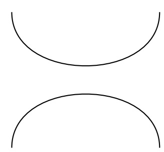
$\mathcal{N}^{(i)} \equiv$  (8)

By replacing sums over Kraus operators with caps, the output of the quantum switch of $\mathcal{N}^{(1)}$ and $\mathcal{N}^{(2)}$ can then be written as a sum of diagrams, each of which we refer to as CPM-like,



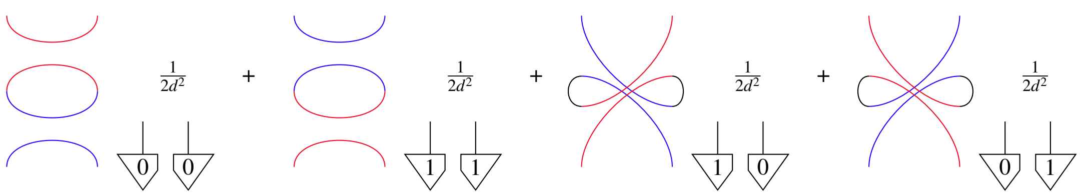
(9)

This picture can be used to reproduce the classical capacity activation in the quantum switch of two CDPC's. We take each of $\mathcal{N}^{(1)}$ and $\mathcal{N}^{(2)}$ to be CDPC's, then in the CPM-construction a CDPC is written

$\mathcal{D} = \frac{1}{d}$  (10)

The diagram for a CDPC separates vertically, which implies that it has no classical or quantum capacity. Upon insertion into equation 9,

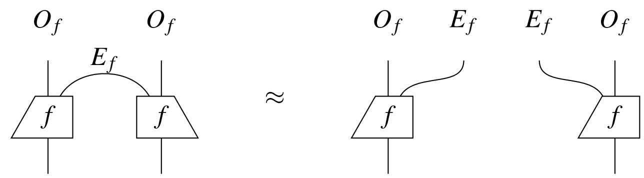
(11)



The above diagrams provide an intuition for capacity activation, as the crossing over of the depolarising channels in the off diagonal components of the control allows information to flow from the input (at the bottom of the picture) to the output (at the top of the picture).

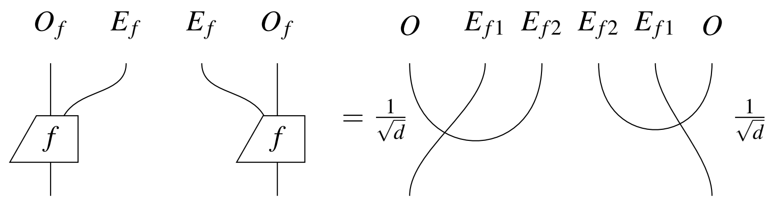
4 Alternative Intuition for Capacity Activation

From the CPM picture, we notice that information flows through the environments of the depolarising channels. Here we use this idea to give an intuition for capacity enhancement. For each channel f with output O_f and environment E_f, rather than working with the CPM representation, we instead keep track of the environment system E_f by working with the Stinespring dilation.



(12)

The depolarising channel, can be dilated as an isometry which sends the input state into an environment and entangles the output with an independent environment,

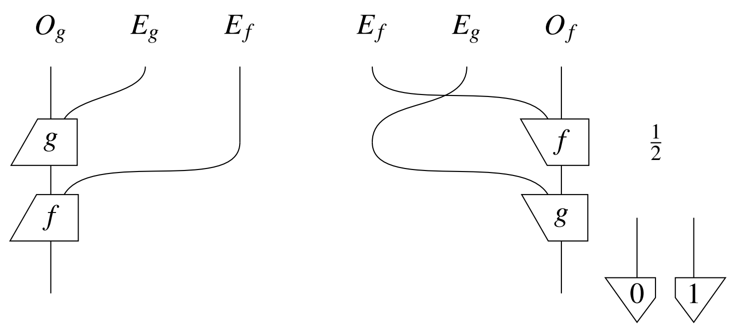


(13)

Where the upwards pointing cup represents a Bell state.

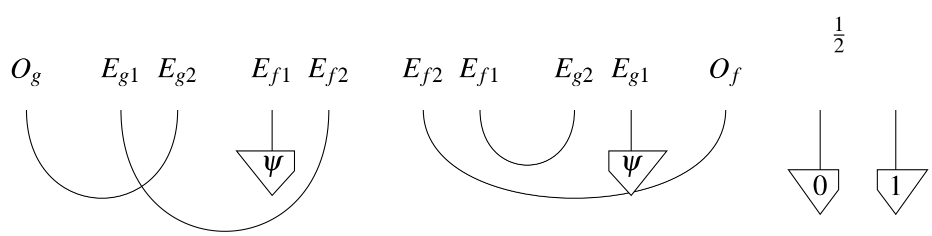
4.1 The Switch of Depolarising Channels as Superposition of the Whereabouts of the Input

Denoting the environments of f and g as E_f and E_g, and their outputs as O_f,O_g, we consider the interference term of the quantum switch of the dilations of f and g



(14)

For f and g dilations of completely depolarising channels this gives



(15)

When a state is inserted into the input of the channel, its whereabouts becomes entangled with the control qubit. In control $|0\rangle$ the state $|\psi\rangle$ is in E_{f1} and in control $|1\rangle$ the state $|\psi\rangle$ is in E_{g1}. The correlations between $|\psi\rangle$ and O_g can be explained in the following way.

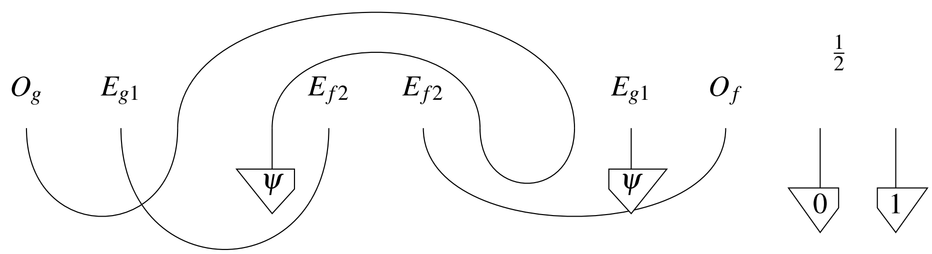
- For control state $|0\rangle$, $|\psi\rangle$ is in environment E_{f1}

- In control state $|1\rangle$, E_{f1} is entangled with E_{g2}
- In control state $|0\rangle$, E_{g_2} is entangled with the output.

For these points to give capacity enhancement requires that

- There are correlations between E_{f1} in branch $|0\rangle$ and E_{f1} in branch $|1\rangle$
- There are correlations between E_{g2} in branch $|0\rangle$ and E_{g2} in branch $|1\rangle$

This is true by virtue of each of these systems really being the same physical system. By ignoring the environments E_{f1} and E_{g2} we forget any knowledge of E_{f1} and E_{g2} and retain only the inescapable fact that they are the same system in each branch of the superposition.

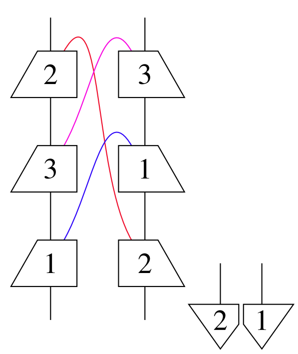


(16)

Tracing out E_{f2} and E_{g1} indeed gives the interference term of the quantum switch.

5 N-Party Switches

A natural generalisation of the quantum switch is coherent control of some choice of sequential orders of 3 or more channels [15]. We could imagine for example using states $|0\rangle, |1\rangle, |2\rangle$ of a control qutrit to implement the sequential compositions $(\mathcal{N}^{(1)} \circ \mathcal{N}^{(2)} \circ \mathcal{N}^{(3)})$, $(\mathcal{N}^{(3)} \circ \mathcal{N}^{(1)} \circ \mathcal{N}^{(2)})$, and $(\mathcal{N}^{(2)} \circ \mathcal{N}^{(3)} \circ \mathcal{N}^{(1)})$ respectively. Again one would expect to see interference between the two choices of sequential order in the left and right hand sides of an interference term of the control,



(17)

Figure 1: CPM-like diagram for the term $|2\rangle\langle 1|$ in the control of a switch which in state $|2\rangle$ implements $\mathcal{N}^{(2)} \circ \mathcal{N}^{(3)} \circ \mathcal{N}^{(1)}$, and in state $|1\rangle$ implements $\mathcal{N}^{(3)} \circ \mathcal{N}^{(1)} \circ \mathcal{N}^{(2)}$. Each colored wire represents a sum over Kraus operators.

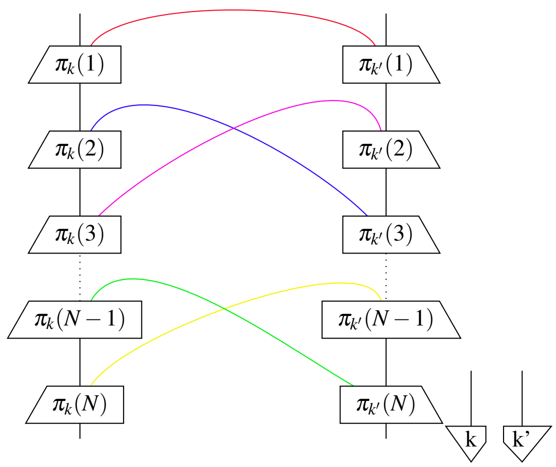
We refer to such a generalisation as a 3-party switch. The advantages for classical capacity enhancement of 3-party switches were explored in [25] where it was observed that the maximum possible Holevo information [19] achievable with a superposition of 3 sequential orders of 3 CDPC's exceeds the maximal Holevo information achievable with 4 or 5 sequential orders of 3 CDPC's. The Holevo information for 3 orders of 3 CDPS's turns out in [25] to be maximised when the orders chosen are cyclic permutations, an observation which we will give good reason to expect to generalise to N channels.

5.1 Diagrammatic Presentation of an N-Party Switch

We call a supermap which generalises the quantum switch to coherent control of M sequential compositions of N channels $\{\mathcal{N}^{(i)}\}_{i=1}^{N}$ each with Kraus operators $\{K_{\mathbf{i}_a}^{(i)}\}_{a=1}^{d^2}$ an N-party switch. If in each of M computational basis states $\{|k\rangle\}_{k=0}^{M-1}$ of a quMit control a supermap would implement one of M permutations $\mathcal{N}^{(\pi_k(N))}\circ\cdots\circ\mathcal{N}^{(\pi_k(1))}$ of the sequential composition $\mathcal{N}^{(N)}\circ\cdots\circ\mathcal{N}^{(1)}$, then for a Fourier state control $|+\rangle_M \equiv \frac{1}{M}\sum_k |k\rangle$ the supermap would implement an equal weighted superposition of each of the M sequential compositions.

$$\begin{aligned}
\rho' &= S_M(\{\mathcal{N}^{(i)}\}_{i=1}^{M})(\rho, |+\rangle\langle+|) \\
&= \frac{1}{M}\sum_{kk'}\sum_{\mathbf{1}_a}^{d^2}\cdots\sum_{\mathbf{N}_a}^{d^2} K_{\pi_\mathbf{k}(\mathbf{1})a}^{(\pi_k(1))}\cdots K_{\pi_\mathbf{k}(\mathbf{N})a}^{(\pi_k(N))}\,\rho\, K_{\pi_{\mathbf{k}'}(\mathbf{N})a}^{(\pi_{k'}(N))^\dagger}\cdots K_{\pi_{\mathbf{k}'}(\mathbf{1})a}^{(\pi_{k'}(1))^\dagger}\otimes|k\rangle\langle k'| \\
&\equiv \frac{1}{M}\sum_{kk'}\mathcal{N}_{kk'}\otimes|k\rangle\langle k'|
\end{aligned} \tag{18}$$

Each k labels a permutation π_k on the order of the channels. As shown in figure 2, diagrammatically the term $\mathcal{N}_{kk'}\otimes|k\rangle\langle k'|$ of the switch is a CPM-like diagram with boxes rearranged according to permutations π_k and $\pi_{k'}$ on the left and right hand wires respectively. Each wire in a CPM-like diagram represents a sum over Kraus operators.

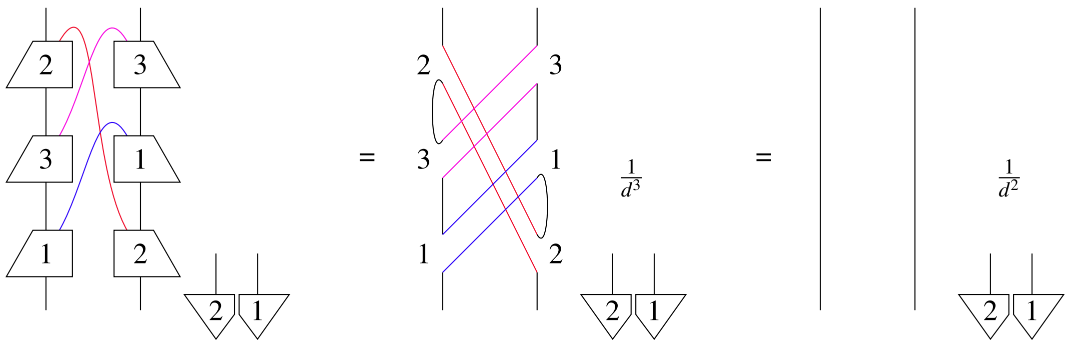


(19)

Figure 2: CPM-like diagram for the term $\mathcal{N}_{kk'}\otimes|k\rangle\langle k'|$. Each cap algebraically is a sum over Kraus operators for a particular channel, as such the above diagram corresponds to the case in which $\pi_k(1) = \pi_{k'}(1)$, $\pi_k(2) = \pi_{k'}(3)$, $\pi_k(3) = \pi_{k'}(2)$, $\pi_k(N-1) = \pi_{k'}(N)$, $\pi_k(N) = \pi_{k'}(N-1)$

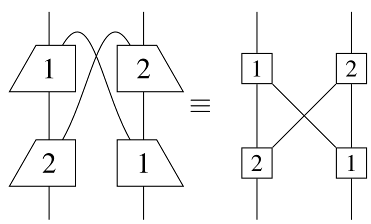
5.2 Capacity Enhancement By Superposition of Cyclic Permutations

For any term $\mathcal{N}_{kk'}$ with π_k and $\pi_{k'}$ cyclic permutations, $\pi_{kk'} \equiv \pi_{k'} \circ \pi_k^{-1}$ is a cyclic permutation, π_k,$\pi_{k'}$ are mutually cyclic. Taking each channel $\mathcal{N}^{(i)}$ to be a completely depolarising channel, it is quick to see by hand in the 3-party case that any such term $\mathcal{N}_{kk'}$ is proportional to an identity channel. For example, for $\mathcal{N}_{21}$



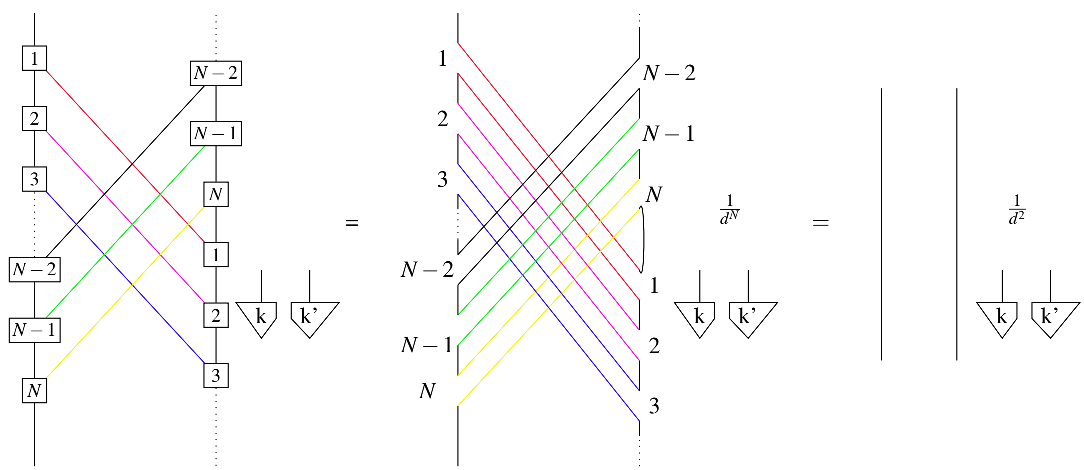
(20)

The normalisation of this information transmitting term is increased by the presence of a closed loop which contributes a factor of d to the diagram. This result immediately generalises, to demonstrate this we first adopt a cleaner notation to cope with the increasing number of boxes



(21)

Then figure 3 presents a generic diagram for a cyclic permutation between left and right hand wires.



(22)

Figure 3: CPM-like Interference diagram for a cyclic permutation between left and right wires is $\frac{1}{d^N}\mathcal{I}$ multiplied by $N-2$ closed loops, giving $\mathcal{N}_{kk'} = \frac{1}{d^2}\mathcal{I}$

Each interference term between cyclic permutations is an information transmitting term with $N-2$ closed loops ensuring that the prefactor of such a term is always $\frac{1}{d^2}$. Since $\mathcal{N}_{kk'}(\rho)$ gives $\frac{\rho}{d^2}$ when $k \neq k'$ and $\frac{I}{d}$ when $k = k'$ the superposition of the N cyclic permutations of N channels has a simple algebraic expression

$$S_{NC}(\{\mathcal{N}^{D(i)}\}_{i=1}^{i=N})(\rho, |+\rangle\langle+|) = \sum_i \frac{I}{Nd} Tr(\rho) \otimes |i\rangle\langle i| + \sum_{i\neq j} \frac{\rho}{Nd^2} \otimes |i\rangle\langle j| \tag{23}$$

For a general choice of M permutations $\{\pi_k\}$, it is not true that for all k,k' $\mathcal{N}_{kk'}(\rho) \propto \rho$, and even when there exist k,k' with $\mathcal{N}_{kk'}(\rho) \propto \rho$ it is often true that $\mathrm{tr}[\mathcal{N}_{kk'}] < \frac{1}{d^2}$. Guided by an intuition that the capacity enhancement will be greatest when the number and normalisation of information transmitting terms $\mathcal{N}_{kk'}$ is maximised, we expect the output channel for a general superposition of permutations would then have lower capacity than the case for which all k,k' are mutually cyclic permutations, this intuition correctly predicts the highest capacity superpositions of $N = 3$ CDPC's as explored in [25]. Letting the number of terms $\mathcal{N}_{kk'}$ proportional to the identity channel and the completely depolarising channel be n_{Id} and n_{Dp} respectively, for maximal capacity enhancement we suggest choosing the M orders which optimize

$$\mathscr{O}(S) \equiv \frac{n_{id}E_{Id}}{n_{Dp}E_{Dp}} \tag{24}$$

where E_{Id}, E_{Dp} are the expected value of the normalisation $E(\mathrm{tr}[\mathcal{N}_{kk'}])$ for identity and depolarising terms. In section 5.3, we give a general expression for any interference term $\mathcal{N}_{kk'}$, proving that cyclic permutations uniquely maximise $\mathscr{O}(S)$ for fixed $M \leq N$. This suggests that good candidates for high capacity activation of N CDPC's given a quMit control should consist of superpositions of mutually cyclic permutations.

5.3 Characterising $\mathcal{N}_{kk'}$ by Permutation Properties

Generalising beyond the cyclic case we derive a simple condition on the permutations π_k and $\pi_{k'}$ which can be used to completely determine any $\mathcal{N}_{kk'}$. Firstly π_k and $\pi_{k'}$ can be used to define cycle permutations C_k and $C_{k'}$ by

$$C_k \equiv (0\pi_k(N)\pi_k(N-1)\dots\pi_k(1)) \tag{25}$$

$$C_{k'} \equiv (0\pi_k(1)\dots\pi_k(N-1)\pi_k(N)) \tag{26}$$

We will show that the interference diagram for $\mathcal{N}_{kk'}$ can be used to compute the product

$$C_{kk'} \equiv C_{k'}^{-1} \circ C_k = (0\pi_{k'}(N)\pi_{k'}(N-1)\dots\pi_{k'}(1))(0\pi_k(1)\dots\pi_k(N-1)\pi_k(N)) \tag{27}$$

and crucially we show the converse, that any term $\mathcal{N}_{kk'}$ can be computed by finding the cycle decomposition of $C_{kk'}$. As a corollary we will have demonstrated that the $\mathcal{N}_{kk'}$ are characterised by a cds sortability [3] condition between π_k and $\pi_{k'}$. We write $c_{kk'}$ for the number of cycles in the cycle decomposition of $C_{kk'}$, $\mathcal{D}$ for the completely depolarising channel, and $\mathcal{I}$ for the identity channel.

Theorem 1 (Information Transmission by Cycle Decomposition)**.** *For the term $\mathcal{N}_{kk'}$ in the quantum switch of M orders of N CDPC's,*

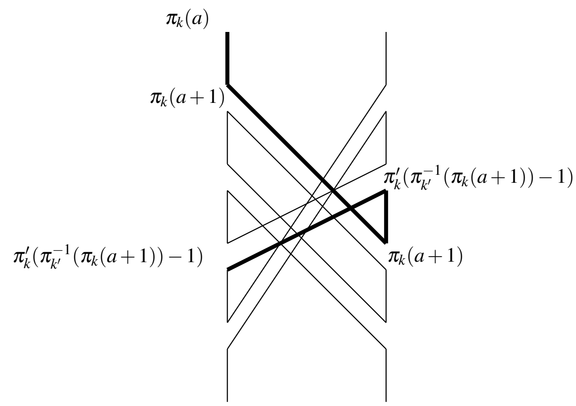
- $\mathcal{N}_{kk'} \propto d\mathcal{D} \iff 0, \pi_k(N)$ *are not in the same cycle of* $C_{kk'}$

- *$\mathcal{N}_{kk'} \propto \mathcal{I} \iff 0, \pi_k(N)$ are in the same cycle of $C_{kk'}$*

Proof. The permutation

$$C_{kk'} \equiv C_{k'}^{-1} \circ C_k = (0\pi_{k'}(N)\pi_{k'}(N-1)\ldots\pi_{k'}(1))(0\pi_k(1)\ldots\pi_k(N-1)\pi_k(N)) \tag{28}$$

Is the function $C_{kk'}(\pi_k(a)) = \pi_{k'}(\pi_{k'}^{-1}(\pi_k(a+1))-1)$. Using the CPM-like diagram for $\mathcal{N}_{kk'}$ in figure 4, the following steps compute $C_{kk'}(\pi_k(a))$ by following a connected path along the diagram from label $\pi_k(a)$ at position a.



(29)

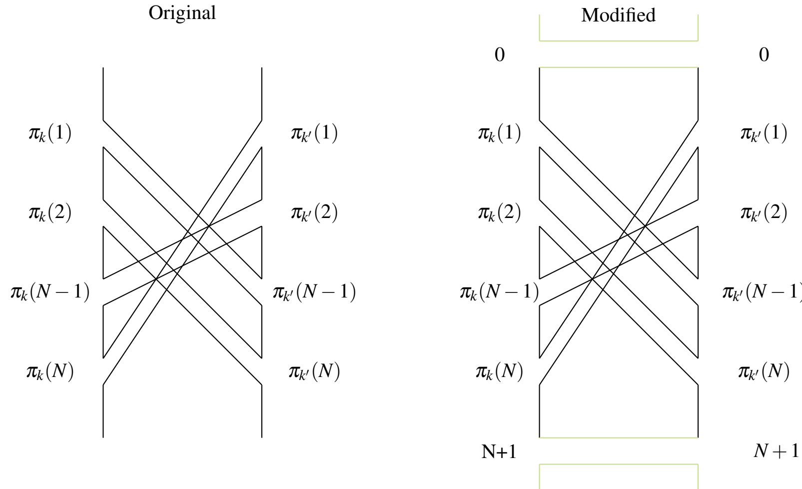
Figure 4: CPM-like diagram used to implement $C_{kk'}$. Being located a slots from the top of the diagram, label π_a is at position a on the LHS

- Start with label $\pi_k(a)$ at position a on the LHS
- Move downwards to label $\pi_k(a+1)$ at position $a+1$
- Use wire to move to the same label $\pi_k(a+1)$ on the RHS, this will be at position $\pi_{k'}^{-1}(\pi_k(a+1))$
- Move up to the label $\pi_{k'}(\pi_k^{-1}(\pi_k(a+1))-1)$ at position $\pi_{k'}^{-1}(\pi_k(a+1))-1$ on the RHS
- Use wire to move to move to same label $\pi_{k'}(\pi_k^{-1}(\pi_k(a+1))-1)$ on the LHS

These steps implement $C_{kk'}(\pi_k(a))$ except for when the steps require a path which is undefined due to the open ends of the CPM-like digram, I.E when $\pi_k(a+1) = 0$ or $a = N$. The modification in figure 5 of the CPM-like diagram accounts for these edge cases and so can be used to compute $C_{kk'}(\pi_k(a))$ for any a. If by starting at label 0 on the LHS, iterating the above steps reaches node $\pi_k(N)$ on the LHS, the diagram is connected from top left to the bottom left, the same will be true for the RHS since the unmodified diagram can have no other open ends, and the diagram will be proportional to the identity channel. Iteration of the above steps is repeated application of $C_{kk'}$, it follows that if in the cycle decomposition of $C_{kk'}$, 0 and $\pi_k(N)$ are in the same cycle, then $\mathcal{N}_{kk'}$ is proportional to the identity channel. Alternatively if 0 and $\pi_k(N)$ are not in the same cycle the channel is proportional to the completely depolarising channel. □

Furthermore the cycle decomposition of $C_{kk'}$ completely determines the normalisation of each $\mathcal{N}_{kk'}$.

Theorem 2 (Normalisation by Cycle Decomposition)**.** *The normalisation of $\mathcal{N}_{kk'}$ is determined by the number of cycles in the cycle decomposition of $C_{kk'}$*



(30)

Figure 5: Modification of CPM-like diagram so that it may be used to fully evaluate $C_{kk'}$

- $\mathcal{N}_{kk'} = \frac{1}{d^N} d^{c_{kk'}-2} d\mathcal{D}$ *when* 0 *and* $\pi_k(N)$ *are not in the same cycle of* $C_{kk'}$
- $\mathcal{N}_{kk'} = \frac{1}{d^N} d^{c_{kk'}-1}\mathcal{I}$ *when when* 0 *and* $\pi_k(N)$ *are in the same cycle of* $C_{kk'}$

Proof. Given in Appendix B. □

In [3] it is proved that π_k and $\pi_{k'}$ are cds sortable iff 0 and $\pi_{k'}(N)$ are in the same cycle of $C_{kk'}$.

Corollary 3 (Information Transmission by CDS Sortability)**.** *The term* $\mathcal{N}_{kk'}$ *is*

- *Proportional to the completely depolarising channel if* π_k *and* $\pi_{k'}$ *are cds sortable*
- *Proportional to the Identity channel if* π_k *and* $\pi_{k'}$ *are not cds sortable*

It is also shown in [3] that for any k,k' with π_k not cds sortable to $\pi_{k'}$, $C_{kk'}$ has maximal number of cycles in its cycle decomposition if and only if $\pi_{k'}\pi_k^{-1}$ is a cyclic permutation.

Corollary 4 (The Cyclic Permutation Protocol Optimises $\mathscr{O}(S)$ for $M \leq N$)**.**

Proof. Given in Appendix C. □

6 Summary

By translating an algebraic expression for the quantum switch into a sum of diagrams we have found an alternative intuition for capacity activation which simply extends to the N cyclic permutations of N channels. Using diagrammatic methods we derived a condition for separability of a diagram in terms of cds sortability which we in turn used to demonstrate optimality of the cyclic protocol with respect to a simple heuristic for information transmission.

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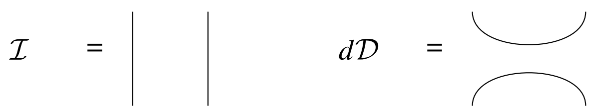
A Proof of Theorem 2

We show that the normalisation of $\mathcal{N}_{kk'}$ is determined by the number of cycles in the cycle decomposition of $C_{kk'}$, specifically

- $\mathcal{N}_{kk'} = \frac{1}{d^N} d^{c_{kk'}-2} d\mathcal{D}$ when 0 and $\pi_k(N)$ are not in the same cycle of $C_{kk'}$
- $\mathcal{N}_{kk'} = \frac{1}{d^N} d^{c_{kk'}-1} \mathcal{I}$ when when 0 and $\pi_k(N)$ are in the same cycle of $C_{kk'}$

Proof. Given in Appendix A. Each substitution of a CDPC into a CPM-like diagram contributes a factor of $\frac{1}{d}$, substitution of all N CDPC's then contributes $\frac{1}{d^N}$. The proportionality constant between $\mathcal{N}_{kk'}$ and each of the diagrams in figure 6 is computed by counting the number of closed loops in the corresponding CPM-like diagram, each of which contributes an additional factor of d.



(31)

Figure 6: We find the proportionality constant between $\mathcal{N}_{kk'}$ and either $\mathcal{I}$ or $d\mathcal{D}$

The number of cycles $c_{kk'}$ in the cycle decomposition of $C_{kk'}$ is the number of closed loops in the modified diagram for $\mathcal{N}_{kk'}$. For $\mathcal{N}_{kk'} \propto d\mathcal{D}$ the modification of the diagram has introduced 2 new closed loops, whereas for $\mathcal{N}_{kk'} \propto \mathcal{I}$ the modification has introduced only 1 extra closed loop into the diagram. As such the number of closed loops in the unmodified CPM-like diagram for $\mathcal{N}_{kk'}$ is

- $c_{kk'} - 2$ when $\mathcal{N}_{kk'} \propto d\mathcal{D}$
- $c_{kk'} - 1$ when $\mathcal{N}_{kk'} \propto \mathcal{I}$

and so the proportionality constant for $\mathcal{N}_{kk'}$ is given by $\frac{1}{d^N} d^{c_{kk'}-2}$ for $\mathcal{N}_{kk'} \propto d\mathcal{D}$ and $\frac{1}{d^N} d^{c_{kk'}-1}$ for $\mathcal{N}_{kk'} \propto \mathcal{I}$. □

B Proof of Corollary 4

We show that the Cyclic Permutation Protocol Optimises $\mathscr{O}(S)$ for $M \leq N$

Proof. Any protocol with M permutations, for which there exists k, k' with $\pi_{k'}\pi_k^{-1}$ not a cyclic permutation, will have some term $|k\rangle\langle k'|$ with either $\mathcal{N}_{kk'}(\rho) = \alpha\frac{\rho}{d^2}$ and $\alpha < 1$ or $\mathcal{N}_{kk'}(\rho) = \beta\frac{I}{d}$. In either case

$$\mathscr{O}(S) < \frac{M(M-1)\frac{1}{d^2}}{M} = \mathscr{O}(S_{\text{cyclic}})$$

□