Partial Boolean algebras and the logical exclusivity principle

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Kochen and Specker's seminal work on contextuality used the formalism of partial Boolean algebras. Unlike quantum logic in the sense of Birkhoff–von Neumann, partial Boolean algebras only admit physically meaningful operations. We describe a refinement of current approaches to contextuality, in particular the sheaf-theoretic and graph-theoretic approaches, to incorporate partial Boolean algebras. We introduce a new axiom for partial Boolean algebras, the Logical Exclusivity Axiom, and show that all probability models based on partial Boolean algebras satisfying this axiom obey Specker's Probabilistic Exclusivity Principle. We also use this principle to give a logical presentation of tensor product which captures some of its salient quantum features.

1   Introduction

Kochen and Specker's seminal work on contextuality used the formalism of partial Boolean algebras [17]. Unlike quantum logic in the sense of Birkhoff and von Neumann [6], partial Boolean algebras only admit physically meaningful operations. In the key example of $\mathsf{P}(\mathscr{H})$, the projectors on a Hilbert space $\mathscr{H}$, the operation of conjunction, i.e. product of projectors, becomes a partial one, only defined on **commuting** projectors. In [16], it is shown that a large part of the foundations of quantum theory can be developed in terms of partial Boolean algebras. In [5], every partial Boolean algebra is shown to be the colimit of its (total) Boolean subalgebras. Thus the topos approach to quantum theory [15] can be seen as a refinement, in explicitly categorical language, of the partial Boolean algebra approach.

In this paper, we relate partial Boolean algebras to current approaches to contextuality, in particular the sheaf-theoretic [3] and graph-theoretic [8] approaches. A major role in our technical development is played by a general result on a universal construction for partial Boolean algebras, which freely generates a new partial Boolean algebra from a given one and extra commeasurability constraints. This result is proved constructively, using an inductive presentation by generators and relations. It provides a flexible tool, subsuming a number of other constructions: some previously known, and some new.

We describe a construction of partial Boolean algebras from graphical measurement scenarios, i.e. measurement scenarios whose measurement compatibility structure is given by a binary compatibility relation, such that empirical models on the scenarios correspond bijectively to probability valuations (or states) on the corresponding partial Boolean algebras.

We discuss how probabilistic contextuality can be formulated in the setting of partial Boolean algebras, and show that the strong state-independent form of contextuality considered by Kochen and Specker can be neatly captured using the universal construction mentioned above.

We also consider to what extent the tensor product operation on Hilbert spaces can be "tracked" by a corresponding operation on partial Boolean algebras. It is easily seen that the construction described in [16] fails to capture all the relations holding in the partial Boolean algebra of projectors on the Hilbert space tensor product. This motivates us to propose a new principle, the **Logical Exclusivity Principle**, which is always satisfied in partial Boolean algebras of the form $\mathsf{P}(\mathscr{H})$. We show that there is a reflection between partial Boolean algebras and those satisfying the Logical Exclusivity Principle.

Our main results relate this Logical Exclusivity Principle to Specker's Exclusivity Principle for probabilistic models [7]. We show that if a partial Boolean algebra satisfies the Logical Exclusivity Principle, then all its states satisfy the Probabilistic Exclusivity Principle. Moreover, we show that a state on a partial Boolean algebra satisfies the Probabilistic Exclusivity Principle if it extends to one on its reflection, i.e. the freely generated partial Boolean algebra satisfying the Logical Exclusivity Principle.

We also show that there is a natural monoidal structure on partial Boolean algebras satisfying Logical Exclusivity. This contrasts with the situation for standard contextuality models satisfying Specker's Exclusivity Principle, which are not closed under tensor product.

We then consider the problem of fully capturing the Hilbert space tensor product in logical form, in terms of partial Boolean algebras. We conclude with a discussion of some natural questions which arise from our results.

2 Partial Boolean algebras

2.1 Basic definitions

A partial Boolean algebra A is given by a set (also written A), constants 0, 1, a reflexive, symmetric binary relation $\odot$ on A, read as "commeasurability" or "compatibility", a total unary operation $\neg$, and partial binary operations $\wedge$, $\vee$ with domain $\odot$.

These must satisfy the following property: every set S of pairwise-commeasurable elements must be contained in a set T of pairwise-commeasurable elements which forms a (total) Boolean algebra[1]under the restrictions of the given operations.

Morphisms of partial Boolean operations are maps preserving commeasurability, and the operations wherever defined. This gives a category **pBA**.

In [5, Theorem 4], it is shown that every partial Boolean algebra is the colimit, in **pBA**, of the diagram $\mathscr{C}(A)$ consisting of its Boolean subalgebras and the inclusions between them.

2.2 Colimits and free extensions of commeasurability

In [5], it is shown that **pBA** is cocomplete. Coproducts have a simple direct description. The coproduct $A \oplus B$ of partial Boolean algebras A, B is their disjoint union with 0_A identified with 0_B, and 1_A identified with 1_B. Other than these identifications, no commeasurability holds between elements of A and elements of B. By contrast, coequalisers, and general colimits, are shown to exist in [5] by an appeal to the Adjoint Functor Theorem. One of our technical contributions is to give a direct construction of the needed colimits, by an inductive presentation.

More generally, we use this approach to prove the following result, which freely generates from a given partial Boolean algebra a new one where prescribed additional commeasurability relations are enforced between its elements.

Theorem 1. *Given a partial Boolean algebra A and a binary relation $\odot$ on A, there is a partial Boolean algebra $A[\odot]$ such that:*

- *There is a **pBA**-morphism $\eta \colon A \longrightarrow A[\odot]$ such that $a \odot b \ \Rightarrow \ \eta(a) \odot_{A[\odot]} \eta(b)$.*

- *For every partial Boolean algebra B and **pBA**-morphism $h \colon A \longrightarrow B$ such that $a \odot b \ \Rightarrow \ h(a) \odot_B$*

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[1]From now on, whenever we say just "Boolean algebra", we mean total Boolean algebra.

Table 1: Rules for free partial Boolean algebra with extended compatibility relation

$$\frac{a \in A}{\iota(a)\!\downarrow} \qquad \frac{a \odot_A b}{\iota(a) \odot \iota(b)} \qquad \frac{a \circledcirc b}{\iota(a) \odot \iota(b)}$$

$$\overline{0 \equiv \iota(0_A),\ 1 \equiv \iota(1_A)} \qquad \frac{a \odot_A b}{\iota(a) \wedge \iota(b) \equiv \iota(a \wedge_A b),\ \iota(a) \vee \iota(b) \equiv \iota(a \vee_A b)} \qquad \overline{\neg\iota(a) \equiv \iota(\neg_A a)}$$

$$\overline{0\!\downarrow,\ 1\!\downarrow} \qquad \frac{t \odot u}{t \wedge u\!\downarrow,\ t \vee u\!\downarrow} \qquad \frac{t\!\downarrow}{\neg t\!\downarrow}$$

$$\frac{t\!\downarrow}{t \odot t,\ t \odot 0,\ t \odot 1} \qquad \frac{t \odot u}{u \odot t} \qquad \frac{t \odot u,\ t \odot v,\ u \odot v}{t \wedge u \odot v,\ t \vee u \odot v} \qquad \frac{t \odot u}{\neg t \odot u}$$

$$\frac{t\!\downarrow}{t \equiv t} \qquad \frac{t \equiv u}{u \equiv t} \qquad \frac{t \equiv u,\ u \equiv v}{t \equiv v} \qquad \frac{t \equiv u,\ u \odot v}{t \odot v}$$

$$\frac{t(\vec{x}) \equiv_{\mathsf{Bool}} u(\vec{x}),\ \bigwedge_{i,j} v_i \odot v_j}{t(\vec{v}) \equiv u(\vec{v})} \qquad \frac{t \equiv t',\ u \equiv u',\ t \odot u}{t \wedge u \equiv t' \wedge u',\ t \vee u \equiv t' \vee u'} \qquad \frac{t \equiv u}{\neg t \equiv \neg u}$$

$h(b)$, *there is a unique* **pBA***-morphism* $\hat{h}: A[\circledcirc] \longrightarrow B$ *such that*

$$A \xrightarrow{\ \eta\ } A[\circledcirc]$$
$$h \searrow \quad \downarrow \hat{h}$$
$$B$$

We do not require that the relation $\circledcirc$ include the commeasurability relation $\odot_A$ already defined on A. Of course, it is the case that $A[\circledcirc] \cong A[\odot_A \cup \circledcirc]$ for any $\circledcirc$, but it will be notationally convenient to allow an arbitrary relation $\circledcirc$ in this construction. In particular, note that $A[\varnothing] \cong A[\odot_A] \cong A$.

As already mentioned, this result is proved constructively, by giving proof rules for commeasurability and equivalence relations over a set of syntactic terms generated from A. In fact, we start with a set of "pre-terms" and also give rules for definedness.

First, we define the set of **pre-terms** P inductively, to be the closure of the set of generators

$$G := \{\iota(a) \mid a \in A\}$$

under the Boolean operations and constants. The standard theory of Boolean algebras gives us an equational theory $\equiv_{\mathsf{Bool}}$ for terms in the Boolean signature $\{0, 1, \wedge, \vee, \neg\}$ over the set of generators G together with variables $x, y, \ldots$. Pre-terms are simply the ground terms in this theory. We have the usual notion of substitution of terms for variables: if $t(\vec{x})$ is a term in the variables $x_1, \ldots, x_n$, and if $u_1, \ldots, u_n$ are pre-terms, then $t(\vec{u})$ is the pre-term which results from replacing x_i by u_i for all $i \in \{1, \ldots, n\}$.

We now define a predicate $\downarrow$ (definedness or existence), and binary relations $\odot, \equiv$ on P, by the set of rules in Table 1.

One can show by rule induction that if $t \odot u$ or $t \equiv u$ can be derived, then $t\!\downarrow$ and $u\!\downarrow$ can be derived; and if $t \equiv u$ can be derived, then $t \odot u$ can be derived. To illustrate the first rule on the last line, consider the distributivity axiom: $x \wedge (y \vee z) \equiv_{\mathsf{Bool}} (x \wedge y) \vee (x \wedge z)$. Under the assumptions $t \odot u, u \odot v, t \odot v$, we can infer $t \wedge (u \vee v) \equiv (t \wedge u) \vee (t \wedge v)$.

We define the set of terms $T := \{t \in P \mid t\!\downarrow\}$. The relation $\equiv$ is an equivalence relation on T, by the rules on the fifth line. We define a structure $A[\circledcirc]$ as follows. The carrier is $T/\!\equiv$. The relation $\odot$ is

defined by: $[t] \odot [u] :\Leftrightarrow \vdash t \odot u$, where $\vdash$ means derivability of an assertion from the rules. This is well defined due to the last rule on the fifth line. The operations are defined by representatives: if $[t] \odot [u]$, then $[t] \wedge [u] := [t \wedge u]$, etc. These are shown to be well defined using the congruence rules on the last line.

Proposition 2. $A[\odot]$ *is a partial Boolean algebra.*

There is a map $\eta : A \longrightarrow A[\odot]$ which sends a to $[\iota(a)]$. This is a **pBA**-morphism which moreover satisfies $a \odot b \Rightarrow \eta(a) \odot \eta(b)$, by the rules on the first two lines.

Now given a partial Boolean algebra B, and a morphism $h : A \longrightarrow B$ such that $a \odot b \Rightarrow h(a) \odot_B h(b)$, we shall show that there is a unique partial Boolean algebra morphism $\hat{h} : A[\odot] \longrightarrow B$ such that

$$A \xrightarrow{\ \eta\ } A[\odot]$$
$$h \searrow \quad \downarrow \hat{h}$$
$$B$$

Let α be one of the assertions $t\downarrow$, $t \odot u$, $t \equiv u$. If Π is a proof of α by the rules in Table 1 with last rule $\frac{\alpha_1, \dots \alpha_n}{\alpha}$, where α_i has proof Π_i, we define the **height** of Π by $\mathsf{ht}(\Pi) := 1 + \max_i \mathsf{ht}(\Pi_i)$. For $n \geq 0$, we define $\vdash_n \alpha$ to mean that there is a proof of α of height at most n. We write $\vdash_{<n} \alpha$ to mean that $\vdash_m \alpha$ for some $m < n$.

Lemma 3. *For all pre-terms t, u:*

 1. $\vdash_n t \odot u$ implies $\vdash_{<n} t\downarrow$ and $\vdash_{<n} u\downarrow$.

 2. $\vdash_n t \equiv u$ implies $\vdash_{<n} t\downarrow$ and $\vdash_{<n} u\downarrow$ and $\vdash_{<n} t \odot u$.

We define a partial map $\varphi : P \rightharpoonup B$ by structural recursion on pre-terms:

$$\varphi(\iota(a)) := h(a)$$
$$\varphi(t \wedge u) := \varphi(t) \wedge_B \varphi(u)$$
$$\varphi(t \vee u) := \varphi(t) \vee_B \varphi(u)$$
$$\varphi(\neg t) := \neg_B \varphi(t)$$

Note that this map is partial because the operations $\wedge_B$, $\vee_B$ are.

Proposition 4. *For all $n \geq 0$ and pre-terms t, u, the following conditions hold:*

 1. $\vdash_n t\downarrow \Rightarrow \varphi(t)$ is defined

 2. $\vdash_n t \odot u \Rightarrow \varphi(t) \odot_B \varphi(u)$

 3. $\vdash_n t \equiv u \Rightarrow \varphi(t) = \varphi(u)$.

Proof. Let $\Phi(n)$ be the conjunction of the above three properties. We shall prove $\forall n. \Phi(n)$ by complete induction on n. Given a proof witnessing $\vdash_n \alpha$, we look at the last rule. The assumptions α_i of the rule will have proofs witnessing $\vdash_{<n} \alpha_i$, to which the induction hypothesis applies. It now suffices to verify that each rule is a valid statement about the partial Boolean algebra B when t, u are replaced by $\varphi(t)$, $\varphi(u)$.

For example, consider the third rule on the fourth line (transitivity of $\equiv$). This translates to

$$\frac{\varphi(t) = \varphi(u), \ \varphi(u) = \varphi(v)}{\varphi(t) = \varphi(v)}$$

Using Lemma 3, we have that $\varphi(t) = b$, $\varphi(u) = c$, $\varphi(v) = d$, for well-defined elements $b, c, d \in B$. Then the rule simply expresses transitivity of equality:

$$(b = c) \wedge (c = d) \Rightarrow (b = d).$$

The other cases are similar. $\square$

Proof of Theorem 1

Proof. We can now establish the required universal property. We define $\hat{h}([t]) := \varphi(t)$, and it follows straightforwardly from the definition of φ together with the previous proposition that this has the required properties. □

This result will prove to be very useful in what follows.

Coequalisers and colimits

A variation of this construction is also useful, where instead of just forcing commeasurability, one forces equality. Given a partial Boolean algebra A and a relation $\odot$ as before, we write $A[\odot, \equiv]$ for the algebra generated by the above inductive construction, with one additional rule:

$$\frac{a \odot a'}{\iota(a) \equiv \iota(a')}$$

We can define a **pBA**-morphism $\eta : A \longrightarrow A[\odot, \equiv]$ by $\eta(a) := [\iota(a)]$. Clearly this satisfies $a \odot a' \Rightarrow \eta(a) = \eta(a')$. A simple adaptation of the proof of Theorem 1 establishes the following universal property of this construction.

Theorem 5. *Let $h : A \longrightarrow B$ be a **pBA**-morphism such that $a \odot a' \Rightarrow h(a) = h(a')$. Then there is a unique **pBA**-morphism $\hat{h} : A[\odot, \equiv] \longrightarrow B$ such that $h = \hat{h} \circ \eta$.*

This result can be used to give an explicit construction of coequalisers, and hence general colimits, in **pBA**. Given a diagram

$$A \underset{g}{\overset{f}{\rightrightarrows}} B$$

in **pBA**, we define a relation $\odot$ on B by $b \odot b' := \exists a \in A . f(a) = b \wedge g(a) = b'$. Then, $\eta : B \longrightarrow B[\odot, \equiv]$ is the coequaliser of f and g.

2.3   States

Definition 6. A **state** or **probability valuation** on a partial Boolean algebra A is a map $\nu : A \longrightarrow [0, 1]$ such that:

1. $\nu(0) = 0$;

2. $\nu(\neg x) = 1 - \nu(x)$;

3. for all $x, y \in A$ with $x \odot y$, $\nu(x \vee y) + \nu(x \wedge y) = \nu(x) + \nu(y)$.

Proposition 7. *States can be characterised as the maps $\nu : A \longrightarrow [0, 1]$ such that, for every Boolean subalgebra B of A, the restriction of ν to B is a finitely additive probability measure on B.*

Lemma 8. *Let A be a partial Boolean algebra. There is a one-to-one correspondence between:*

- *states on A;*

- *families $(\nu_S)_{S \in \mathscr{C}(A)}$ indexed by the Boolean subalgebras S of A, where ν_S is a finitely additive probability measure on S and $\nu_S = \nu_T \circ \iota_{S,T}$ whenever $S \subseteq T$.*

Lemma 9. *Let A be a finite Boolean algebra. There is a one-to-one correspondence between states on A and probability distributions on its set of atoms.*

Proof. Write X for the set of atoms of A. Let $v \colon A \longrightarrow [0,1]$ be a state on A. Then,

$$\sum_{x \in X} v(x) = v(\bigvee X)$$

can be shown by induction on the size of X by using Definition 6–1 for the base case, and Definition 6–1&3 and the fact that $x \wedge y = 0$ when x and y are atoms for the induction step. Since $\bigvee X = 1$, we conclude that $\sum_{x \in X} v(x) = 1$ and so $v|_X \colon X \longrightarrow [0,1]$ is a probability distribution on X.

Conversely, if $d \colon X \longrightarrow [0,1]$ is a probability distribution, we can extend it to the whole Boolean algebra, using the fact that any element of the algebra can be written as a join of atoms, as follows: for any $S \subseteq X$,

$$v(\bigvee S) = \sum_{x \in S} d(x) \, .$$

$\square$

3   Graphical measurement scenarios and partial Boolean algebras

3.1   Measurement scenarios and (no-signalling) empirical models

We consider the basic framework of the sheaf-theoretic approach introduced in [3] to provide a unified perspective on non-locality and contextuality. Our main focus here will not be on the question of contextuality, but rather on principles that approximate the set of quantum-realisable behaviours.

Measurement scenarios provide an abstract notion of an experimental setup. They model a situation where there is a set of measurements, or queries, one can perform on a system, but not all of which may be performed simultaneously.

In this paper, we focus on what we term 'graphical' scenarios, where a subset of measurements is compatible (i.e. can be performed together) if its elements are pairwise compatible. Hence, compatibility is specified simply by a binary relation. A paradigmatic example is quantum theory, where compatibility is given by commutativity: a set of measurements (observables) can be performed together if and only if its elements commute pairwise.

Note that, in contrast to [3], we do not require that the set of measurements be finite. We do, however, consider only measurements with a finite set of outcomes. This allows us to include within the scope of our discussion the scenario formed by all the quantum-mechanical measurements on a system described by a finite-dimensional Hilbert space.

Definition 10. A **graphical measurement scenario** is a triple $\langle X, \frown, O \rangle$ consisting of:

- a set X of measurements,

- a reflexive, symmetric relation $\frown$ on X, indicating compatibility of measurements.

- a family $(O_x)x \in X$ assigning a finite set O_x of outcomes to each measurement $x \in X$.

A **context** is a subset of measurements, $\sigma \subset X$, which are pairwise compatible, i.e. a clique of the relation $\frown$. We write $\mathsf{Kl}(\frown)$ for the set of contexts.

A particular case of interest is that of measurement scenarios where every measurement is dichotomic, i.e. has two possible outcomes.

Given a measurement scenario, an empirical model specifies particular probabilistic observable behaviour that may be displayed by a physical system.

Definition 11. Let $\langle X, \frown, O \rangle$ be a measurement scenario. A (no-signalling) **empirical model** is a family $(e_\sigma)_{\sigma \in \mathsf{Kl}(\frown)}$ where for each context $\sigma \in \mathsf{Kl}(\frown)$, e_σ is a probability distribution on the set $\mathscr{E}(\sigma) := \prod_{x \in \sigma} O_x$ of joint assignments of outcomes to the measurements in σ, and such that $e_\sigma = e_\tau|_\sigma$ whenever $\sigma, \tau \in \mathsf{Kl}(\frown)$ with $\sigma \subseteq \tau$, where $e_\tau|_\sigma$ is marginalisation of distributions given by: for any $\mathbf{s} \in \mathscr{E}(\sigma)$,

$$e_\tau|_\sigma(\mathbf{s}) := \sum_{\mathbf{t} \in \mathscr{E}(\tau), \mathbf{t}|_\sigma = \mathbf{s}} e_\tau(\mathbf{t}) \,.$$

Moreover, such an empirical model is said to be **non-contextual** if there is a (global) probability distribution d on the set $\mathscr{E}(X) = \prod_{x \in X} O_x$ which marginalises to the empirical probabilities, i.e. such that $d|_\sigma = e_\sigma$ for all contexts $\sigma \in \mathsf{Kl}(\frown)$.

The marginalisation condition in the definition of empirical models ($e_\sigma = e_\tau|_\sigma$ for contexts σ, τ) ensures that the probabilistic outcome of a compatible subset of measurements is independent of which other compatible measurements are performed alongside these. This is sometimes referred to as the **no-disturbance condition** [20], or **no-signalling condition** [19] in the special case of Bell scenarios. This is a local compatibility condition, whereas non-contextuality can be seen as global compatibility: this justifies the slogan that contextuality arises from empirical data which is locally consistent but globally inconsistent.

No-disturbance is satisfied by any empirical probabilities that can be realised in quantum mechanics [3]. However, this condition is much weaker than quantum realisability. Empirical models allow for behaviours that may be considered super-quantum, exemplified by the PR box [19]. A lot of effort has gone into trying to characterise the set of quantum behaviours by imposing some additional, physically motivated conditions on empirical models, leading to various approximations from above to this quantum set.

3.2   Exclusivity principle on empirical models

One candidate for a property which is distinctive for the quantum case has appeared in various formulations as Local Orthogonality [9], Consistent Exclusivity [13], or Specker's Exclusivity Principle [7]. We shall refer to it as the Probabilistic Exclusivity Principle (PEP), since it is expressed as a constraint on probability assignments.

Informally, it says that if we have a family of pairwise exclusive events, the sum of their probabilities must sum to ≤ 1. Of course, if all the events live on a single sample space, this would just be a basic property of probability measures. What gives the condition its force is that these events live, in general, over different, **incompatible** contexts. Thus it reaches beyond the usual view of contexts as different classical "windows" on a quantum system, in which incompatible contexts are regarded as incommensurable.

We can give a precise formulation of PEP in terms of empirical models as follows. First, we say that events $s \in \mathscr{E}(\sigma)$, $t \in \mathscr{E}(\tau)$ are **exclusive** if for some $x \in \sigma \cap \tau$, $s(x) \neq t(x)$. Now the principle holds for an empirical model $(e_\sigma)_\sigma$ if, whenever we have a family $\{s_i \in \mathscr{E}(\sigma_i)\}_{i \in I}$ of pairwise-exclusive events, then

$$\sum_i e_{\sigma_i}(s_i) \leq 1.$$

This principle is valid in quantum-realisable empirical models, in which measurements correspond to observables, because incompatible (non-commuting) observables can share projectors, and exclusivity of outcomes with respect to common projectors implies **orthogonality**.

Although we know that PEP does not fully characterise the quantum-realisable empirical models, it stands as an important and fruitful principle [13, 4]. We wish to study this principle from the perspective of partial Boolean algebras.

3.3   From graphical measurement scenarios to partial Boolean algebras

To any graphical measurement scenario, we can associate a partial Boolean algebra whose states correspond to empirical models.

Definition 12. Let $\mathbf{X} = \langle X, \frown, O \rangle$ be a graphical measurement scenario. We define its associated partial Boolean algebra $A_{\mathbf{X}}$ as follows:

1. For each measurement $x \in X$, we write B_x for the atomic Boolean algebra with atoms corresponding to O_x. We write $[x = o]$ for the atom of B_x corresponding to the outcome $o \in O_x$.

2. We consider the partial Boolean algebra $A := \bigoplus_{x \in X} B_x$, the coproduct of all the Boolean algebras B_x taken in the category **pBA**. Note that all its elements are of the form $\iota_x(a)$ for a unique $x \in X$ and $a \in B_x$, except for the constants 0 and 1.

3. We define the following relation $\odot$ on the elements of A:

$$\iota_x(a) \odot \iota_y(b) \qquad \text{iff} \qquad x \frown y \text{ or } a \in \{0,1\} \text{ or } b \in \{0,1\}$$

4. Take $A_{\mathbf{X}} := A[\odot]$, the extension of A by the relation $\odot$, as given by Theorem 1.

We can give an alternative description using colimits.

Definition 13. Let $\mathbf{X} = \langle X, \frown, O \rangle$ be a graphical measurement scenario. We define an associated partial Boolean algebra $B_{\mathbf{X}}$ as follows:

- for each measurement $x \in X$, we write B_x for the atomic Boolean algebra with atoms corresponding to O_x. We write $[x = o]$ for the atom of B_x corresponding to the outcome $o \in O_x$.

- for each context $\sigma \in \mathsf{Kl}(\frown)$, we write $B_\sigma := \sum_{x \in \sigma} B_x$, the coproduct of all the B_x with $x \in \sigma$ in the category **BA** of Boolean algebras.[2]

- given contexts $\sigma, \tau \in \mathsf{Kl}(\frown)$ with $\sigma \subseteq \tau$, there is a Boolean algebra homomorphism $\iota_\sigma^\tau \colon B_\sigma \longrightarrow B_\tau$ given by the obvious injection.

- the partial Boolean algebra $B_{\mathbf{X}}$ is the colimit in **pBA** of the diagram consisting of the Boolean algebras $(B_\sigma)_{\sigma \in \mathsf{Kl}(\frown)}$ and the inclusions $(\iota_\sigma^\tau)_{\sigma \subseteq \tau \in \mathsf{Kl}(\frown)}$.

Note that the colimit in this instance can be given very explicitly. The carrier set of $B_{\mathbf{X}}$ is the union of all the B_σ where σ is a maximal clique of $\frown$, and the Boolean subalgebras of $B_{\mathbf{X}}$ are exactly those in the family $(B_\sigma)_{\sigma \in \mathsf{Kl}(\frown)}$.

Proposition 14. *The two descriptions coincide: $A_{\mathbf{X}} \cong B_{\mathbf{X}}$.*

Proposition 15. *There is a one-to-one correspondence between states on $A_{\mathbf{X}}$ and empirical models on* **X**.

Proof. This follows from the fact that the Boolean subalgebras of $A_{\mathbf{X}}$ are the family $(B_\sigma)_{\sigma \in \mathsf{Kl}(\frown)}$, by applying Lemmas 8 and 9, and noting that the condition $\nu_S = \nu_T \circ \iota_{S,T}$ when $S \subseteq T$ on states on total Boolean subalgebras translates under the correspondence in Lemma 9 to marginalisation of probability distributions.   □

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[2]Note that the set of atoms of this coproduct Boolean algebra is the cartesian product of the sets of atoms of each of the summands. Hence an atom of B_σ corresponds to an assignment of an outcome in O_x to each measurement $x \in \sigma$.

4 Partial Boolean algebras and contextuality

We consider some aspects of contextuality formulated in the framework of partial Boolean algebras, and relate them to the free construction from Theorem 1.

4.1 The Kochen–Specker property

The Kochen–Specker theorem, as originally stated [17], is that there are partial Boolean algebras of Hilbert space projectors with no **pBA**-morphisms to **2**. Since every (non-trivial[3]) Boolean algebra has a homomorphism to **2**, this implies that such a partial Boolean algebra A has no morphism to **any** (non-trivial) Boolean algebra.

  Now, **BA** is a full subcategory of **pBA**. We know from [5] that A is the colimit in **pBA** of the diagram $\mathscr{C}(A)$ consisting of its Boolean subalgebras and inclusions between them. Let B be the colimit in **BA** of the same diagram $\mathscr{C}(A)$. Then, the cone from $\mathscr{C}(A)$ to B is also a cone in **pBA**, hence there is a mediating morphism from A to B.

  To resolve the apparent contradiction, note that **BA** is an equational variety of algebras over **Set**. As such, it is complete and cocomplete, but it also admits the one-element Boolean algebra **1**, in which $0 = 1$. Note that this trivial Boolean algebra **1** does **not** have a homomorphism to **2**.

  We can conclude from the discussion above that a partial Boolean algebra satisfies the Kochen–Specker property of not having a morphism to **2** if and only if the colimit of its diagram of Boolean subalgebras is **1**. In fact, we could formulate this property directly for diagrams of Boolean algebras, without referring to partial Boolean algebras at all: a diagram in **BA** is K–S if its colimit in **BA** is **1**. We could say that such a diagram is "implicitly contradictory", since in trying to combine all the information in a colimit we obtain the manifestly contradictory **1**.

  Finally, this property admits a neat formulation in terms of the free extension of partial Boolean algebras by a relation, reminiscent of the definition of a perfect group.

Theorem 16. *Let A be a partial Boolean algebra. The following are equivalent:*

  *1. A has the K-S property, i.e. it has no morphism to **2**.*

  *2. The colimit in **BA** of the diagram $\mathscr{C}(A)$ of Boolean subalgebras of A is **1**.*

  3. $A[A^2] = \mathbf{1}$.

Proof. The equivalence between the first two statements follows from the discussion above. Now, all elements are commeasurable in $A[A^2]$, so it is a Boolean algebra. There is a morphism $A \longrightarrow \mathbf{2}$ if and only if there is a morphism $A[A^2] \longrightarrow \mathbf{2}$, by the universal property of $A[A^2]$ (in the $\Rightarrow$ direction) or composition with $\eta : A \longrightarrow A[A^2]$ (in the $\Leftarrow$ direction). Since $A[A^2]$ is a Boolean algebra, this is in turn equivalent to $A[A^2]$ being non-trivial. In other words, there is no morphism $A \longrightarrow \mathbf{2}$ if and only if $A[A^2] = \mathbf{1}$. $\qquad\square$

4.2 Probabilistic contextuality

The notion of contextuality for state also admits a formulation in this setting.

Definition 17. A state $\nu : A \longrightarrow [0,1]$ on a partial Boolean algebra A is said to be **non-contextual** if it extends to $A[A^2]$, i.e. there is a state $\hat{\nu} : A[A^2] \longrightarrow [0,1]$ sich that $\nu = \hat{\nu} \circ \eta$.

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[3]See the following discussion.

By the universal property of $A[A^2]$, this is equivalent to asking that there is some Boolean algebra B, a morphism $h\colon A \longrightarrow B$, and state $\hat{v}\colon B \longrightarrow [0,1]$ such that $v = \hat{v} \circ \eta$.

Proposition 18. *Let* **X** *be a graphical measurement scenario. A state on* $\mathfrak{A}_{\mathbf{X}}$ *is contextual in the sense of Definition 17 if and only if the corresponding empirical model under the correspondence of Proposition 15 is contextual in the sense of Definition 11.*

Note that if A has the Kochen–Specker property, we have $A[A^2] = \mathbf{1}$, and since there is no state on $\mathbf{1}$, every state of A is contextual. An advantage of partial Boolean algebras is that the K-S property provides an intrinsic, logical approach to defining **state-independent contextuality**.

5   Exclusivity principles for partial Boolean algebras

We now consider exclusivity principles from the perspective of partial Boolean algebras. This will subsume the previous discussion on PEP for empirical models in graphical measurement scenarios.

We introduce two exclusivity principles: one acts at the 'logical' level, i.e. the level of events or elements of a partial Boolean algebra, whereas the other acts at the 'probabilistic' level, applying to states of a partial Boolean algebra.

5.1   Logical exclusivity principle

The basic ingredient is a notion of exclusivity between events (or elements) of a partial Boolean algebra. Given a partial Boolean algebra A and $a, b \in A$, we write $a \leq b$ to mean that $a \odot b$ and $a \wedge b = a$. Note that the restriction of this relation $\leq$ to any Boolean subalgebra of A coincides with the partial order underlying that Boolean algebra.

Definition 19. Let A be a partial Boolean algebra. Two elements $a, b \in A$ are said to be **exclusive**, written $a \perp b$, if there is a $c \in A$ such that $a \leq c$ and $b \leq \neg c$.

Note that $a \perp b$ is a weaker requirement than $a \wedge b = 0$, although the two would be equivalent in a Boolean algebra. The point is that, in a general partial Boolean algebra, one might have exclusive events that are not commeasurable (and for which, therefore, the $\wedge$ operation is not defined).

Definition 20. A partial Boolean algebra is said to satisfy the **logical exclusivity principle (LEP)** if any two elements that are logically exclusive are also commeasurable, i.e. if $\perp \subseteq \odot$.

We write **epBA** for the full subcategory of **pBA** whose objects are partial Boolean algebras satisfying LEP.

Logical exclusivity and transitivity

The logical exclusivity principle turns out be equivalent to the following notion of transitivity [18, 14].

Definition 21. A partial Boolean algebra is said to be **transitive** if whenever $a \leq b$ and $b \leq c$ holds, then $a \leq c$ holds.

Transitivity can fail in general in a partial Boolean algebra, since we need not have $a \odot c$ under the stated hypotheses. Note that the relation $\leq$ on a partial Boolean algebra is always reflexive and antisymmetric, so this condition is equivalent to $\leq$ being a partial order (globally) on A. Note that a partial Boolean algebra of the form $\mathsf{P}(\mathscr{H})$ is always transitive.

Proposition 22. *Let A be a partial Boolean algebra. Then A satisfies LEP if and only if it is transitive.*

Proof. Suppose that A satisfies LEP, $a \leq b$, and $b \leq c$. Then $\neg c \leq \neg b$. Hence, by LEP, $a \odot \neg c$, and so $a \odot \neg\neg c = c$. Now, $a \wedge c = a \wedge b \wedge c = a \wedge b = a$, showing that $a \leq c$.

Conversely, suppose that A is transitive, $a \leq c$, and $b \leq \neg c$. Then, $c = \neg\neg c \leq \neg b$, hence $a \leq \neg b$ by transitivity. In particular, $a \odot \neg b$, and so $a \odot \neg\neg b = b$. $\qquad\square$

It is shown in [12] that a partial Boolean algebra is transitive if and only if it is an orthomodular poset.

5.2 Probabilistic exclusivity principle

We now consider an analogous principle applying at the probabilistic level, i.e. at the level of states of a partial Boolean algebra.

Definition 23. Let A be a partial Boolean algebra. A state $v \colon A \longrightarrow [0,1]$ on A is said to satisfy the **probabilistic exclusivity principle (PEP)** if for any set $S \subseteq A$ of pairwise exclusive elements, i.e. such that $\forall a, b \in S. \ (a = b \vee a \perp b)$, then $\sum_{a \in S} v(a) \leq 1$.

A partial Boolean algebra is said to satisfy PEP if all of its states satisfy PEP.

Note that the condition $\sum_{a \in S} v(a) \leq 1$ is true of any set S of elements in a Boolean algebra satisfying $\forall a, b \in S. \ a \wedge b = 0$

Note that this subsumes the discussion of the PEP at the level of empirical models. If $\mathbf{X}$ is a measurement scenario, the correspondence in Proposition 15 between empirical models on $\mathbf{X}$ and states of $A_{\mathbf{X}}$ restricts to a bijection between empirical models and states satisfying the probabilistic exclusivity principle.

5.3 LEP *vs* PEP

The following result follows immediately from the definitions of partial Boolean algebras and states.

Proposition 24. *Let A be a partial Boolean algebra satisfying the logical exclusivity principle. Then, any state on A satisfies the probabilistic exclusivity principle.*

In a general partial Boolean algebra A, however, not all states need satisfy the PEP. A well-known example is the state on the partial Boolean algebra corresponding to a $(4,2,2)$ Bell scenario[4] which corresponds to two (independent) copies of the PR box [9].

However, using the construction from Theorem 1, we can construct from A a new partial Boolean algebra, namely $A[\perp]$, whose states yield states of A that satisfy PEP.

Theorem 25. *Let A be a partial Boolean algebra. Then a state $v \colon A \longrightarrow [0,1]$ satisfies PEP if there is a state $\hat{v}$ of $A[\perp]$ such that*

$$
\begin{array}{ccc}
A & \xrightarrow{\ \eta\ } & A[\perp] \\
& {\scriptstyle v}\searrow & \downarrow{\scriptstyle \hat{v}} \\
& & [0,1]
\end{array}
$$

commutes.

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[4]This stands for a scenario in which there are 4 parties, each of which can choose to perform one of 2 measurements with 2 possible outcomes.

Proof. Let $\nu\colon A \longrightarrow [0,1]$ be a state. Suppose it factorises through a state $\hat{\nu}$ of $A[\bot]$. Let $S \subseteq A$ be a set of pairwise exclusive events in A. Then $\{\eta(a) \mid a \in S\}$ is a commeasurable subset of $A[\bot]$, hence it is contained in a Boolean subalgebra B of $A[\bot]$. Since $\hat{\nu}$ must restrict to a finitely-additive probability measure on B, and since $\forall a, b \in S.\ \eta(a) \wedge_{A[\bot]} \eta(b) = 0$, we have that

$$\sum_{a \in S} \nu(a) = \sum_{a \in S} \hat{\nu}(\eta(a)) \leq 1 \ .$$

$\square$

5.4 A reflective adjunction for logical exclusivity

It is not clear whether the partial Boolean algebra $A[\bot]$ necessarily satisfies LEP. While the principle holds for all its elements in the image of $\eta : A \longrightarrow A[\bot]$, it may fail to hold for other elements in $A[\bot]$.

However, we can adapt the construction of Theorem 1 to show that one can freely generate, from any given partial Boolean algebra, a new partial Boolean algebra satisfying LEP. This LEP-isation is analogous to e.g. the way one can 'abelianise' any group, or use Stone–Čech compactification to form a compact Hausdorff space from any topological space.

Theorem 26. *The category* **epBA** *is a reflective subcategory of* **pBA**, *i.e. the inclusion functor* $I\colon$ **epBA** $\longrightarrow$ **pBA** *has a left adjoint* $X\colon$ **pBA** $\longrightarrow$ **epBA**. *Concretely, to any partial Boolean algebra A, we can associate a partial Boolean algebra* $X(A) = A[\bot]^*$ *which satisfies LEP such that:*

- *there is a homomorphism* $\eta\colon A \longrightarrow A[\bot]^*$;

- *for any homomorphism* $h\colon A \longrightarrow B$ *where B is a partial Boolean algebra B satisfying LEP, there is a unique homomorphism* $\hat{h}\colon A[\bot]^* \longrightarrow B$ *such that:*

$$A \xrightarrow{\ \eta\ } A[\bot]^*$$
$$h \searrow \quad \downarrow \hat{h}$$
$$B$$

The proof of this result follows from a simple adaptation of the proof of Theorem 1, namely adding the following rule to the inductive system presented in Table 1:

$$\frac{u \wedge t \equiv u,\ v \wedge \neg t \equiv v}{u \odot v}$$

This rule will enforce the logical exclusivity principle, and the universal property is proved in a manner similar to the proof of Theorem 1.

6 Tensor products of partial Boolean algebras

6.1 A (first) tensor product by generators and relations

In [5], it is shown that **pBA** has a monoidal structure, with $A \otimes B$ given by the colimit of the family of $C + D$, as C ranges over Boolean subalgebras of A, D ranges over Boolean subalgebras of B, and $C + D$ is the coproduct of Boolean algebras.

The tensor product in [5] is not constructed explicitly: it relies on the existence of coequalisers in **pBA**, which is proved by an appeal to the Adjoint Functor Theorem.

Our Theorem 1 allows us to give an explicit description of this construction using generators and relations.

Proposition 27. *Let A and B be partial Boolean algebras. Then*

$$A \otimes B \;\cong\; (A \oplus B)[\circlearrowright]$$

where $\circlearrowright$ is the relation on the carrier set of $A \oplus B$ given by $\iota(a) \circlearrowright \jmath(b)$ for all $a \in A$ and $b \in B$.

This can be verified by comparing the universal property from Theorem 1 with [5, Proposition 30].

6.2   A more expressive tensor product

There is a lax monoidal functor $\mathsf{P} \colon \mathbf{Hilb} \longrightarrow \mathbf{pBA}$, which takes a Hilbert space to its projectors, viewed as a partial Boolean algebra, with a morphism $\mathsf{P}(\mathscr{H}) \otimes \mathsf{P}(\mathscr{K}) \longrightarrow \mathsf{P}(\mathscr{H} \otimes \mathscr{K})$ induced by the evident embeddings of $\mathsf{P}(\mathscr{H})$ and $\mathsf{P}(\mathscr{K})$ into $\mathsf{P}(\mathscr{H} \otimes \mathscr{K})$), given by $p \longmapsto p \otimes 1$, $q \longmapsto 1 \otimes q$.

It is easy to see that this morphism is far from being an isomorphism. For example, if we take $\mathscr{H} = \mathscr{K} = \mathbb{C}^2$, then there are (many) two-valued homomorphisms on $A = \mathsf{P}(\mathbb{C}^2)$, which lift to two-valued homomorphisms on $A \otimes A$. However, by the Kochen–Specker theorem, there is no such homomorphism on $\mathsf{P}(\mathbb{C}^4) = \mathsf{P}(\mathbb{C}^2 \otimes \mathbb{C}^2)$.

Interestingly, in [16] it is shown that the images of $\mathsf{P}(\mathscr{H})$ and $\mathsf{P}(\mathscr{K})$, for any finite-dimensional $\mathscr{H}$ and $\mathscr{K}$, generate $\mathsf{P}(\mathscr{H} \otimes \mathscr{K})$. This is used in [16] to justify the claim contradicted by the previous paragraph. The gap in the argument is that more relations hold in $\mathsf{P}(\mathscr{H} \otimes \mathscr{K})$ than in $\mathsf{P}(\mathscr{H}) \otimes \mathsf{P}(\mathscr{K})$. Nevertheless, this result is very suggestive. In standard Boolean algebra theory, these images would satisfy the criteria for being an "internal direct sum" of $\mathsf{P}(\mathscr{H})$ and $\mathsf{P}(\mathscr{K})$ [10]. Evidently, for partial Boolean algebras, these criteria are no longer sufficient. This poses the challenge of finding stronger criteria, and a stronger notion of tensor product to match.

An important property satisfied by the rules in Table 1 as applied in constructing $A \otimes B$ is that, if $t\!\downarrow$ can be derived, then $u\!\downarrow$ can be derived for every subterm u of t. This appears to be too strong a constraint to capture the full logic of the Hilbert space tensor product.

To see why this is an issue, consider projectors $p_1 \otimes p_2$ and $q_1 \otimes q_2$. To ensure in general that they commute, we need the conjunctive requirement that p_1 commutes with q_1, **and** p_2 commutes with q_2. However, to show that they are **orthogonal**, we have a disjunctive requirement: $p_1 \perp q_1$ **or** $p_2 \perp q_2$. If we establish orthogonality in this way, we are entitled to conclude that $p_1 \otimes p_2$ and $q_1 \otimes q_2$ are commeasurable, even though (say) p_2 and q_2 are not. Indeed, the idea that propositions can be defined on quantum systems even though subexpressions are not is emphasized in [16].

This leads us to define a stronger tensor product by forcing logical exclusivity to hold in the tensor product from [5]. This amounts to composing with the reflection to **epBA**; $\boxtimes := X \circ \otimes$. Explicitly, we define the logical exclusivity tensor product by

$$A \boxtimes B = (A \otimes B)[\perp]^* = (A \oplus B)[\circlearrowright][\perp]^*.$$

This is sound for the Hilbert space model. More precisely, P is still a lax monoidal functor with respect to this tensor product. It remains to be seen how close it gets us to the full Hilbert space tensor product.

6.3   Commeasurability extensions, Kochen–Specker, and Hilbert space tensor product

We can ask generally if extending commeasurability by some relation R can induce the Kochen–Specker property in $A[R]$ when it did not hold in A. In fact, it is easily seen that this can never happen.

Theorem 28 (K-S faithfulness of extensions)**.** *Let A be a partial Boolean algebra, and $R \subseteq A^2$ a relation on A. Then A has the K-S property if and only if $A[R]$ does.*

Proof. If A does not have the K-S property, it has a homomorphism to a non-trivial Boolean algebra B. By the universal property of $A[R]$, there is a homomorphism $\hat{h} \colon A[R] \longrightarrow B$. Thus, $A[R]$ does not have the K-S property. Conversely, if there is a morphism $k \colon A[R] \longrightarrow B$ to a non-trivial Boolean algebra B, then $k \circ \eta \colon A \longrightarrow B$, so A does have the K-S property.   □

We can apply this in particular to the tensor product.

Corollary 29. *If A and B do not have the K-S property, then neither does $A \otimes B[\bot]^k$.*

Proof. If A and B do not satisfy the K-S property, they have homomorphisms to **2**, and hence so does $A \oplus B$. Applying Theorem 28 inductively $k+1$ times, so does $A \otimes B[\bot]^k = A \oplus B[\oplus][\bot]^k$.   □

Under the conjecture that $A[\bot]^*$ coincides with iterating $A[\bot]$ to a fixpoint, this would show that the logical exclusivity tensor product $A \boxtimes B$ never induces a Kochen–Specker paradox if none was already present in A or B.

This can be seen as a limitative result, in the following sense. One of the key points at which non-classicality emerges in quantum theory is the passage from $\mathsf{P}(\mathbb{C}^2)$, which does not have the K-S property, to $\mathsf{P}(\mathbb{C}^4) = \mathsf{P}(\mathbb{C}^2 \otimes \mathbb{C}^2)$, which does.[5] By contrast, it follows from the Corollary that $\mathsf{P}(\mathbb{C}^2) \boxtimes \mathsf{P}(\mathbb{C}^2)$ does not have the K-S property. Therefore, we need a stronger tensor product to track this emergent complexity in the quantum case.

7   Discussion

There are a number of questions arising from the ideas developed in this paper.

- First, we have shown that LEP implies PEP; that is, if a partial Boolean algebra satisfies Logical Exclusivity, then all its states satisfy Probabilistic Exclusivity. We conjecture that the converse holds.

  Conjecture 30. *PEP $\Rightarrow$ LEP.*

- Similarly, we conjecture the converse to Theorem 25.

  Conjecture 31. *If state ν of a partial Boolean algebra A satisfies PEP, then there is a state $\hat{\nu}$ of $A[\bot]$ such that $\nu = \hat{\nu} \circ \eta$.*

  This would amount to generalising the universality of $A[\bot]$ from **pBA**-homomorphisms to states. It would yield a one-to-one correspondence between states of A satisfying PEP and states of $A[\bot]$.

- Proving the conjecture above would involve extending a state on a partial Boolean algebra A to a state on $A[\odot]$. A similar operation was achieved for partial Boolean algebras arising from measurement scenarios in Proposition 15, because in that case Definition 13 provided a simple description of the Boolean subalgebras of $A[\odot]$. Is an analogous description possible for the general case considered in Theorem 1, or at least for the particular case of $A[\bot]$?

- A classic result by Greechie [11] constructs a class of orthomodular lattices which admit no states. Since orthomodular lattices are transitive partial Boolean algebras (see e.g. [21]), this means there are examples of partial Boolean algebras satisfying LEP which admit no states. Is there a partial Boolean algebra not satisfying LEP which admits no states? This would provide a counter-example to Conjecture 30.

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[5]Note that $\mathsf{P}(\mathbb{C}^2) \cong \bigoplus_{i \in I} \mathbf{4}_i$, where I is a set of the power of the continuum, and each $\mathbf{4}_i$ is the four-element Boolean algebra.

- There are some technical questions relating to the $A[\perp]^*$ construction:
  - Is it a completion (i.e. is the reflector a faithful functor)?
  - Is it the same as iterating the $A[\perp]$ construction to a fixpoint?
  - Is the relation of $A[\perp]^*$ to $A[\perp]$ an instance of a more general relationship between iterating an inductive construction, and adding a rule to the inductive construction itself?

- Our discussion of tensor products led us to introduce a strong tensor product of partial Boolean algebras, $A \boxtimes B$. This brings us closer to an answer to the following particularly interesting question:

  Question 32. Is there a monoidal structure $\circledast$ on **pBA** such that the functor $\mathsf{P} \colon$ **Hilb** $\longrightarrow$ **pBA** is **strong monoidal** with respect to this structure, i.e. such that $\mathsf{P}(\mathscr{H}) \circledast \mathsf{P}(\mathscr{K}) \cong \mathsf{P}(\mathscr{H} \otimes \mathscr{K})$?

  A positive answer to this question would offer a complete logical characterization of the Hilbert space tensor product, and provide an important step towards giving logical foundations for quantum theory in a form useful for quantum information and computation.

- We recall the following quotation from Ernst Specker given in [7]:

  > Do you know what, according to me, is the fundamental theorem of quantum mechanics? ... That is, if you have several questions and you can answer any two of them, then you can also answer all three of them. This seems to me very fundamental.

  This refers to the **binarity** of compatibility in quantum mechanics. A set of observables is compatible if they are pairwise so. This is built in to the definition of partial Boolean algebras, and is why we only considered graphical measurement scenarios in this paper. However, in the general theory of contextuality, as developed e.g. in [3], more general forms of compatibility are considered, represented by simplicial complexes. Can partial Boolean algebras be adapted to this more general format, and how much of the theory carries over?

- Partial Boolean algebras capture logical structure. We have seen how this logical structure can be used to enforce strong constraints on the probabilistic behaviour of states. This is somewhat analogous to the role of possibilistic empirical models in [3]. Can we lift the concepts and results relating to possibilistic empirical models in [3, 2, 1] to the level of partial Boolean algebras?

- There is much more to be said regarding contextuality in this setting. In current work in progress, we are considering the following questions:
  - A hierarchy of logical contextuality properties generalising those studied in [3].
  - A systematic treatment of "Kochen-Specker paradoxes", i.e. contradictory statements which can be validated in partial Boolean algebras.
  - The role played by non-free Boolean subalgebras in constructing such paradoxes.
  - Constructions which transform state-dependent to state-independent forms of contextuality.

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