Quantum predicate logic with equality

Andre Kornell
Department of Computer Science
Tulane University

In 2001, Weaver extended the propositional quantum logic of Birkhoff and von Neumann to a predicate logic, with a natural interpretation of the standard quantifiers. In this submission, we further extend this quantum predicate logic with a natural interpretation of the equality relation. To accommodate this equality relation, we work exclusively with quantum sets, essentially von Neumann algebras of a particularly simple form. The resulting semantics may be used as a uniform method of quantization for discrete structures. We recover the dagger compact category of quantum sets and binary relations, and the monoidal closed category of quantum sets and functions, both introduced in an earlier work of the author. These categories have recently been used to model recursion in the quantum setting. We also recover the standard notions of a quantum graph and a quantum monoid. Finally, we recover winning quantum strategies for the graph coloring game as quantum families of proper graph colorings. (This is an extended abstract for arXiv:2004.04377.)

Let us assume that any physical system can be modeled by a von Neumann algebra. Such a physical system may be fully classical, in the sense that every nontrivial observable is compatible with every other nontrivial observable, or fully quantum, in the sense that every nontrivial observable is incompatible with some other nontrivial observable. Thus, fully classical systems are modeled by commutative von Neumann algebras, and fully quantum systems are modeled by factors.

A physical system may exhibit both classical and quantum behaviors, and such systems are modeled by von Neumann algebras that are neither commutative nor factors. Such systems occur naturally within quantum information theory. For example, a quantum computer with n bits and m qubits is modeled by the von Neumann algebra

$$\underbrace{\mathbb{C}^2 \mathbin{\bar{\otimes}} \cdots \mathbin{\bar{\otimes}} \mathbb{C}^2}_{n} \mathbin{\bar{\otimes}} \underbrace{M_2(\mathbb{C}) \mathbin{\bar{\otimes}} \cdots \mathbin{\bar{\otimes}} M_2(\mathbb{C})}_{m}.$$

In this submission, we do not restrict our attention to finitary physical systems, but rather to discrete physical systems. We say that a physical system is discrete in case each observable admits a complete set of eigenstates. Physically, we mean that every state of the system has a nonzero transition probability to some eigenstate of the observable; mathematically, we mean that the corresponding self-adjoint operator admits an orthonormal basis of eigenvectors. A physical system is discrete in this sense if and only if the corresponding von Neumann algebra is of a very special form.

Proposition 1 (proposition 5.4 of [7])**.** For each von Neumann algebra M, the following are equivalent:

1. Every self-adjoint operator $a \in M$ admits an orthonormal basis of eigenvectors.
2. There is an indexed family of positive integers $(n_i \,|\, i \in I)$ such that $M \cong \bigoplus_{i \in I} M_{n_i}(\mathbb{C})$.

We call such a von Neumann algebra M hereditarily atomic. Hereditarily atomic von Neumann algebras are a quantum generalization of sets [7]. We associate each ordinary set S with the hereditarily atomic von Neumann algebra $\ell^\infty(S)$, which consists of all bounded complex-valued functions on S. Up to isomorphism, a hereditarily atomic von Neumann algebra is of this form if and only if it is commutative. The physical intuition is that the set S is the configuration space of a discrete fully classical system modeled by the von Neumann algebra $\ell^\infty(S)$.

Submitted to:
QPL 2020

We may define a quantum set to be simply a hereditarily atomic von Neumann algebra. Following standard practice in quantum mathematics, we will assiduously avoid saying that a quantum set $\mathcal{X}$ is a hereditarily atomic von Neumann algebra, instead writing $\ell^\infty(\mathcal{X})$ for this von Neumann algebra. Formally, $\ell^\infty(\mathcal{X})$ is simply equal to $\mathcal{X}$, but intuitively, $\ell^\infty(\mathcal{X})$ consists of bounded complex-valued functions on a discrete quantum space $\mathcal{X}$, which is mathematically fictitious. Definition 2.1 of [7] is a technically convenient definition of quantum sets, but it is unnecessary to the expository goals of this abstract. For the simplicity of exposition, we diverge slightly from the notation and terminology of the source [6].

We now recall Weaver's semantics for first-order formulas [8, 2.6] [2], generalizing it to hereditarily atomic von Neumann algebras and to many-sorted formulas, in the obvious way. We diverge from Weaver's approach only in our treatment of duplicate variables. The sorts of our language are quantum sets, and the relation symbols of our language are relations on quantum sets, defined in the expected way. Indeed, the analogy between projection operators and subsets, and the analogy between the spatial tensor product and the Cartesian product are both firmly established within quantum mathematics.

Definition 2. Let $\mathcal{X}_1,\ldots,\mathcal{X}_n$ be quantum sets. A relation of arity $(\mathcal{X}_1,\ldots,\mathcal{X}_n)$ is a projection operator $R \in \ell^\infty(\mathcal{X}_1)\mathbin{\bar{\otimes}}\cdots\mathbin{\bar{\otimes}}\ell^\infty(\mathcal{X}_n)$.

The core of the semantics is the interpretation of primitive formulas. We interpret other first-order formulas by translating them into primitive formulas. An atomic formula is defined to be primitive iff it contains no function symbols and its variables are all distinct. More generally, a first-order formula is defined to be primitive iff it is built up from primitive atomic formulas using just the connectives $\neg$ and $\wedge$, and just the quantifier $\forall$.

Definition 3. The interpretation of primitive formulas Φ may be defined recursively as follows:

1. $[\![R(x_1,\ldots,x_n)]\!] = R$
2. $[\![\neg\Phi(x_1,\ldots,x_n)]\!] = 1 - [\![\Phi(x_1,\ldots,x_n)]\!]$
3. $[\![\Phi(x_1,\ldots,x_n)\wedge\Psi(x_1,\ldots,x_n)]\!] = [\![\Phi(x_1,\ldots,x_n)]\!]\wedge[\![\Psi(x_1,\ldots,x_n)]\!]$
4. $[\![\forall x_1\in\mathcal{X}_1.\,\Phi(x_1,\ldots,x_n)]\!] = \sup\{S\in\mathrm{Proj}(\ell^\infty(\mathcal{X}_2)\mathbin{\bar{\otimes}}\cdots\mathbin{\bar{\otimes}}\ell^\infty(\mathcal{X}_n))\,|\,1\otimes S \le [\![\Phi(x_1,\ldots,x_n)]\!]\}$

For the sake of clarity and brevity, this statement of the definition neglects contexts, which are necessary to exactly determine the interpretation of a formula. For example, the relation $[\![(x,y)\in\mathcal{X}\times\mathcal{Y}\,|\,\Phi(x,y)]\!]$ is a projection in $\ell^\infty(\mathcal{X})\mathbin{\bar{\otimes}}\ell^\infty(\mathcal{Y})$, the relation $[\![(y,x)\in\mathcal{Y}\times\mathcal{X}\,|\,\Phi(x,y)]\!]$ is a projection in $\ell^\infty(\mathcal{Y})\mathbin{\bar{\otimes}}\ell^\infty(\mathcal{X})$, and the relation $[\![(x,y,z)\in\mathcal{X}\times\mathcal{Y}\times\mathcal{Z}\,|\,\Phi(x,y)]\!]$ is a projection in $\ell^\infty(\mathcal{X})\mathbin{\bar{\otimes}}\ell^\infty(\mathcal{Y})\mathbin{\bar{\otimes}}\ell^\infty(\mathcal{Z})$. These details are treated thoroughly in the source [6]. As expected, the disjunction connective $\vee$ is defined to be dual to the conjunction connective $\wedge$, the existential quantifier $\exists$ is defined to be dual to the universal quantifier $\forall$, and the implication connective $\to$ is defined to be the Sasaki arrow [5].

We would define the equality relation for a quantum set $\mathcal{X}$ to be the largest projection $E_\mathcal{X}\in\ell^\infty(\mathcal{X})\mathbin{\bar{\otimes}}\ell^\infty(\mathcal{X})$ such that $E_\mathcal{X}\perp P\otimes(1-P)$ for every projection $P\in\ell^\infty(\mathcal{X})$, but this projection $E_\mathcal{X}$ is equal to zero for qubits $\mathcal{X}$, i.e., when $\ell^\infty(\mathcal{X})\cong M_2(\mathbb{C})$. Instead, we define the equality relation on $\mathcal{X}$ to be the largest projection $E_\mathcal{X}\in\ell^\infty(\mathcal{X})\mathbin{\bar{\otimes}}\ell^\infty(\mathcal{X})^*$ such that $E_\mathcal{X}\perp P\otimes(1-P)^*$ for every projection $P\in\ell^\infty(\mathcal{X})$. The von Neumann algebra $\ell^\infty(\mathcal{X})^*$ consists of dual operators; it is canonically antiisomorphic to $\ell^\infty(\mathcal{X})$. It is natural to speak of a dual quantum set $\mathcal{X}^*$ such that $\ell^\infty(\mathcal{X}^*)\cong\ell^\infty(\mathcal{X})^*$. This equality relation $E_\mathcal{X}$ is nondegenerate for each quantum set $\mathcal{X}$, in the sense that $[\![\forall x_1\in\mathcal{X}.\,\exists x_2\in\mathcal{X}^*.\,E_\mathcal{X}(x_1,x_2)]\!] = 1\in\mathbb{C}$. In fact, we show that this nondegeneracy characterizes the class of hereditarily atomic von Neumann algebras.

Armed with this equality relation, we may in effect simultaneously quantify over a quantum set $\mathcal{X}$ and its dual $\mathcal{X}^*$. This defined quantifier plays an important role in axiomatization and computation.

Definition 4. We write $\forall(x_1=x_2)\in\mathcal{X}\times\mathcal{X}^*.\ \Phi(x_1,\ldots,x_n)$ as an abbreviation for $\forall x_1\in\mathcal{X}.\forall x_2\in\mathcal{X}^*.$ $(x_1=x_2\to\Phi(x_1,\ldots,x_n))$, where $x_1=x_2$ is another notation for $E_\mathcal{X}(x_1,x_2)$.

The importance of this defined quantifier is explained by the fact that the dual quantifier $\exists(x_1 = x_2) \in \mathcal{X} \times \mathcal{X}^*$ corresponds to a contraction in the dagger-compact category **qRel** of quantum sets and the binary relations between them [7]. This dagger-compact category is dual to the category of hereditarily atomic von Neumann algebras and quantum relations in the sense of [9]. Many of our calculations are performed using the graphical calculus for dagger-compact categories, following [1].

The category **qRel** may also be obtained directly from **FdHilb**, by first replacing operators with operator subspaces, and second, replacing objects by tuples of objects, and morphisms by matrices of morphisms, both possibly infinite. Intuitively, the first step replaces transition amplitudes with cruder data about the *possibility* of state transition, and the second step adds sum types. This sum construction is classical in the sense that a pure state on $\ell^\infty(\mathcal{X} \uplus \mathcal{Y}) := \ell^\infty(\mathcal{X}) \oplus \ell^\infty(\mathcal{Y})$ is either a pure state on $\ell^\infty(\mathcal{X})$ or a pure state on $\ell^\infty(\mathcal{Y})$, but not a superposition of the two. It formalizes a control flow by classical data.

The category **qRel** can itself be described within the semantics of this submission. Its subcategory **qSet** of quantum sets and the functions between them [7] can also be described within this semantics.

Proposition 5. Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be quantum sets, let R be a relation of arity $(\mathcal{X}, \mathcal{Y}^*)$, and let S be a relations of arity $(\mathcal{Y}, \mathcal{Z}^*)$. Define $S \circ R$ to be $[\![\exists(y_2 = y_1) \in \mathcal{Y} \times \mathcal{Y}^*. R(x, y_1) \wedge S(y_2, z)]\!]$, a relation of arity $(\mathcal{X}, \mathcal{Z}^*)$. The resulting category is equivalent to **qRel**.

Proposition 6. Let $\mathcal{X}$ and $\mathcal{Y}$ be quantum sets, and let F be a relation of arity $(\mathcal{X}, \mathcal{Y}^*)$. Define F to be a function iff $[\![\forall y_1 \in \mathcal{Y}^*. \forall y_2 \in \mathcal{Y}. (\exists(x_1 = x_2) \in \mathcal{X} \times \mathcal{X}^*. (F(x_1, y_1) \wedge F^*(x_2, y_2)) \to y_2 = y_1)]\!] = 1$, and $[\![\forall x \in \mathcal{X}. \exists y \in \mathcal{Y}^*. F(x, y)]\!] = 1$. The resulting subcategory is dual to the category of hereditarily atomic von Neumann algebras and unital normal †-homomorphisms.

The duality between these two categories allows us to push states forward along functions. Indeed, let F be a function from $\mathcal{X}$ to $\mathcal{Y}$, and let ϕ be the corresponding unital †-homomorphism from $\ell^\infty(\mathcal{Y})$ to $\ell^\infty(\mathcal{X})$. It is natural to define the pushforward of a normal state $\mu \colon \ell^\infty(\mathcal{X}) \to \mathbb{C}$ to be $\mu \circ \phi$.

We close by exhibiting connections between the semantics of this submission, and two examples from quantum information theory. Our first example is quantum graphs, which first appeared as the confusability graphs of quantum channels [4].

Proposition 7. Let $\mathcal{X}$ be a quantum set with $\ell^\infty(\mathcal{X}) \cong M_d(\mathbb{C})$ for some d. There is a canonical bijective correspondence between quantum graphs on $\ell^\infty(\mathcal{X})$ in the sense of [4], and relations R of arity $(\mathcal{X}, \mathcal{X}^*)$ such that $[\![\forall(x_1 = x_2) \in \mathcal{X} \times \mathcal{X}^*. R(x_1, x_2)]\!] = 1$, and $[\![\forall x_1 \in \mathcal{X}. \forall x_2 \in \mathcal{X}^*. R(x_1, x_2) \to R^*(x_2, x_1)]\!] = 1$.

Our second example is the existence of winning quantum strategies for the graph coloring game [3]. The parameters of this game are a graph G, and a set S, intuitively of colors, and the rules are such that classically Alice and Bob have a winning strategy if and only if G may be properly colored by the colors in T. To state the result, we need some extra notation, which will be glossed after the statement.

Proposition 8. There exists a winning quantum strategy for the graph coloring game if and only if there exists a quantum set $\mathcal{X} \neq \emptyset$, and a function $F \colon \mathcal{X} \times G \to S$ such that

$$[\![\forall(x_1 = x_2) \in \mathcal{X} \times \mathcal{X}^*. \forall g_1 \in G. \forall g_2 \in G^*. (g_1 \sim g_2 \to \neg\, F(x_1, g_1) = F^*(x_2, g_2))]\!] = \top.$$

We gloss the new notation as follows. First, each ordinary set S is naturally identified with a quantum set, by demanding that $\ell^\infty(S)$ retain its standard meaning. Thus, G, S and $\emptyset$ may also be regarded as quantum sets. Second, each binary relation on S is naturally identified with a relation in the sense of definition 2, by regarding it as a function $S \times S \to \{0, 1\}$, and therefore a projection in $\ell^\infty(S \times S) \cong \ell^\infty(S) \bar{\otimes} \ell^\infty(S)$. Thus, the adjacency relation $\sim$ may be regarded as a relation of arity (S, S). Because $\ell^\infty(S)$ is canonically isomorphic to $\ell^\infty(S)^*$, the adjacency relation $\sim$ may also be regarded as a relation of arity (S, S^*), as it is in the formula in proposition 8. Finally, the equation $F(x_1, g_1) = F^*(x_2, g_2)$ is taken to abbreviate the formula $\exists(y_1 = y_3) \in T \times T^*. \exists(y_2 = y_4) \in T^* \times T. y_1 = y_2 \wedge F(x_1, g_1, y_3) \wedge F^*(x_2, g_2, y_4)$. The general translation scheme for atomic formulas is given in [6, 2.6.3]. The atomic formula $F(x_1, g_1) = F^*(x_2, g_2)$ is also equivalent to the simpler primitive formula $\exists(y_1 = y_2) \in T^* \times T. F(x_1, g_1, y_1) \wedge F^*(x_2, g_2, y_2)$.

References

[1] S. Abramsky & B. Coecke: *Categorical quantum mechanics*. Available at `arXiv:0808:1023`.

[2] G. Birkhoff & J. von Neumann (1936): *The Logic of Quantum Mechanics*. Ann. Math. 37.

[3] P. J. Cameron, A. Montanaro, M. W. Newman, S. Severini & A. Winter (2007): *On the quantum chromatic number of a graph*. Electron. J. Combin. 14(1).

[4] R. Duan, S. Severini & A. Winter (2013): *Zero-error communication via quantum channels, noncommutative graphs, and a quantum Lovász number*. IEEE Trans. Inform. Theory 59(2).

[5] G. M. Hardegree (1981): *Material Implication in Orthomodular (and Boolean) Lattices*. Notre Dame J. Form. Log. 22(2).

[6] A. Kornell: *Quantum predicate logic with equality*. Available at `arXiv:2004.04377`.

[7] A. Kornell: *Quantum sets*. To appear in J. Math. Phys. Available at `arXiv:1804.00581`.

[8] N. Weaver (2001): *Mathematical Quantization*. Studies in Advanced Mathematics, Chapman & Hall/CRC.

[9] N. Weaver (2012): *Quantum relations*. Mem. Amer. Math. Soc. 215. Available at `arXiv:1005.0354`.