There and back again: A circuit extraction tale*

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Introduction — Procedures for translating between the quantum circuit model and the measurement-based one-way model are useful for verification and optimisation of quantum computations [2, 22]. Existing results require the existence of a *gflow* on the measurement pattern. Each measurement pattern has an underlying graph augmented with some additional information; gflow is a property of this graph which ensures determinism of the pattern. While the concept of gflow is defined for patterns allowing measurements in three different planes of the Bloch sphere, most research so far has focused on patterns containing only measurements in the XY-plane. We give the first efficient circuit-extraction algorithm that works for all patterns that have a gflow, including those with measurements in all three planes. Furthermore, our algorithm results in ancilla-free circuits. We represent measurement patterns using the ZX-calculus and hence our algorithm also represents the most general known procedure for extracting circuits from ZX-diagrams. In developing this algorithm, we generalise and unify several concepts and results previously known for patterns containing only XY-plane measurements. With these results, we can simplify measurement patterns by reducing the number of measured qubits while preserving both the semantics of the pattern and the existence of gflow (and hence the deterministic implementability of the pattern).

Background — The circuit model and the measurement-based model are two fundamentally different approaches to implementing quantum computations. In the circuit model [20], measurements serve mainly to read out data and may often be postponed to the end of the computation, while in the measurement-based model the bulk of the computation is performed via measurements. In the one-way model [21] of measurement-based quantum computing, the starting resource is a *graph state*, and all measurements are performed on single qubits.

Computations in the one-way model are represented by *measurement patterns* [7, 8], which describe both the graph state, the measurements performed on it, and necessary corrections depending on the measurement outcomes. Instead of allowing arbitrary single-qubit measurements, measurements are usually restricted to the planes of the Bloch sphere that are spanned by two of the principal axes, labelled the XY-, XZ-, and YZ-planes. Since quantum measurements are non-deterministic, later measurement angles must be adapted according to the outcomes of earlier measurements in order to achieve an overall deterministic computation [21] . However, not all measurement patterns support such corrections. The existence of a *causal flow* is a sufficient condition for patterns containing only measurements in the XY-plane to be deterministically implementable [6]. This condition is not necessary for determinism, however, and also does not allow measurements in different planes. The broader property of having a *generalized flow* (or *gflow*) [3] is both sufficient and necessary for deterministic implementability. The concept of gflow can be defined for patterns containing measurements in all three planes, in which

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$$M_6^{\mathrm{XY},0} M_2^{\mathrm{XY},0} M_5^{\mathrm{XY},\frac{\pi}{4}} M_1^{\mathrm{XY},\frac{\pi}{4}} M_4^{\mathrm{YZ},-\frac{\pi}{4}} E_{12}E_{14}E_{23}E_{45}E_{56}E_{67}N_2N_3N_4N_6N_7$$

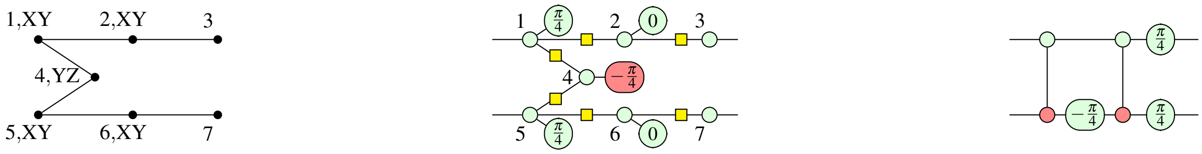

Figure 1: Example of the translation between measurement patterns and circuits via the ZX-calculus. At the top we have a measurement pattern where $M_j^{\lambda,\alpha}$ represents a measurement on qubit j in plane λ at angle α, E_{jk} represents a controlled-Z operation between qubits j and k, and N_j represents the preparation of qubit j. Correction operators are ignored as they will not appear in the ZX-diagram. Below and to the left, we have the underlying labelled open graph of the pattern, where $\{1,5\}$ are inputs and $\{3,7\}$ are outputs. The middle diagram shows the corresponding MBQC-form ZX-diagram and the right-most diagram shows the extracted circuit.

case it is sometimes called *extended gflow* [3, Theorems 2 & 3]. The *underlying labelled open graph* of a measurement pattern specifies a graph, the sets of 'input' and 'output' qubits, and the choice of measurement plane for every non-output qubit. A gflow is defined on such a labelled open graph as a partial order on the qubits, representing the time order in which measurements are to be performed, and a correction set for each qubit, showing which future measurements need to be modified to correct an undesired outcome.

Translation and rewriting — Translations between the quantum circuit model and the one-way model use a type of local rewriting that translates computations piece-by-piece. Local rewriting techniques can also be used to transform computations within the same model, e.g. to find representations with more desirable properties such as a smaller number of qubits, lower computational depth, or simpler operations and resources. The ZX-calculus [4] allows straightforward representations of both measurement patterns and quantum circuits, and it has several complete sets of graphical rewrite rules [10, 13, 14, 23]. This makes it a useful intermediary formalism for translating and transforming computations in the two models, as illustrated in Fig. 1. We define a ZX-diagram to be in *MBQC form* if it is related to a measurement pattern in a specific way, and we say such a diagram has gflow if the underlying labelled open graph has gflow. We derive a number of explicit rewrite rules for MBQC-form ZX-diagrams, including rewrites that involve *local complementations* [16] on the underlying resource graph state. Crucially, we show how a gflow changes when each of our rewrite rules is applied. Since the existence of a gflow is preserved, the pattern remains deterministically implementable. Our rewrite rules unify and formalise several rules that were previously employed in the literature in a more ad-hoc manner, e.g. the pivot-minor transformation [19] or the elimination of Clifford measurements first derived in a different context in Ref. [12].

The rewrite rules serve not only to extract a circuit from a measurement pattern, but also to simplify measurement patterns by reducing the number of qubits involved. Combining the different rules allows us to remove any qubit measured in a Clifford basis, while maintaining deterministic implementability. This shows that the number of qubits needed to perform a measurement-based computation is directly related to the number of non-Clifford operations required for the computation.

We generalise several concepts originally developed for patterns containing only XY-plane measurements to patterns with measurements in multiple planes. In particular, we adapt the definitions of *focused gflow* [17] and *maximally delayed gflow* [18] to the extended gflow case. This allows us to generalise the

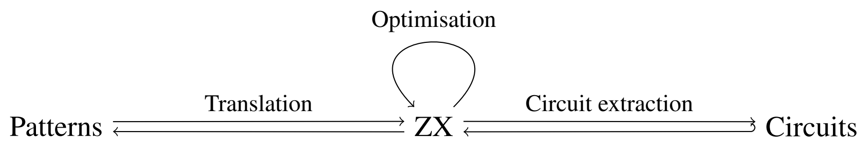

Figure 2: A schematic overview of the translation procedures between the three paradigms.

efficient algorithm of Ref. [18] (which finds a gflow on a given labelled open graph with only XY-plane measurements) to work for patterns with measurements in multiple planes. Note that our generalisation of focused gflow differs from the three generalisations suggested by Hamrit and Perdrix [11].

Circuit extraction — Using the rewrite rules for ZX-diagrams in MBQC-form, we give an algorithm that extracts an ancilla-free quantum circuit from any measurement pattern with extended gflow. This is the first circuit extraction algorithm for extended gflow, i.e. for patterns which contain measurements in more than one plane. The algorithm works by translating the pattern into the ZX-calculus and transforming the resulting ZX-diagram into a circuit-like form. It generalises a similar algorithm, which works only for patterns where all measurements are in the XY-plane [9]. The circuit extraction algorithm employs the ZX-calculus, so it can be used not only on diagrams arising from measurement patterns but on any ZX-diagram that satisfies the MBQC-form properties and has gflow. In particular, this includes the ZX-diagrams arising from the procedures of Refs. [9, 15] that contain *phase gadgets* [1, 5]. Our procedure is therefore not only the most general circuit extraction algorithm for measurement patterns but also the most general known circuit extraction algorithm for ZX-diagrams.

Combined with the known procedure for transforming a quantum circuit into a measurement pattern using the ZX-calculus [9], our pattern simplification and circuit extraction procedure can be used to reduce the non-Clifford gate count of quantum circuits. This involves translating the circuit into a ZX-diagram, transforming to a diagram which corresponds to a measurement pattern, simplifying the pattern, and then re-extracting a circuit. Hence, our results can be seen as a unification of the optimisation of measurement patterns and circuits. See Fig. 2 for a schematic overview of this procedure.

Conclusions — We have generalised several concepts involving gflow to work for patterns involving measurements in multiple planes. This has allowed us to unify multiple known results regarding rewriting measurement patterns. As a result we can remove all internal qubits measured in a Clifford basis from a pattern while preserving deterministic realisability. Furthermore, we can efficiently transform any measurement pattern with gflow into an ancilla-free quantum circuit. This transformation also represents the most general known circuit extraction algorithm for ZX-diagrams.

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