Joint measurability structures realizable with qubit measurements: incompatibility via marginal surgery

Nikola Andrejic
University of Niš, Serbia
Faculty of Sciences and Mathematics, Niš, Serbia
nikola.andrejic@pmf.edu.rs

Ravi Kunjwal
Perimeter Institute for Theoretical Physics
Waterloo, Ontario Canada
Centre for Quantum Information and Communication
Ecole polytechnique de Bruxelles, Brussels, Belgium
rkunjwal@ulb.ac.be

A given set of measurements can exhibit a variety of incompatibility relations and an intuitive way to represent its joint measurability structure is via hypergraphs. The vertices of such a hypergraph denote measurements and the hyperedges denote (all and only) subsets of compatible measurements. While projective measurements in quantum theory can realize (all and only) joint measurability structures that are graphs, general measurements represented by positive operator-valued measures (POVMs) can realize arbitrary joint measurability structures. This latter fact opens up the possibility of studying generalized contextuality à la Spekkens on joint measurability structures not realizable with projective measurements (and which are therefore excluded from traditional treatments of Kochen-Specker contextuality). Here we explore the scope of joint measurability structures that are realizable with POVMs on a qubit. We develop a technique that we term *marginal surgery* to obtain nontrivial joint measurability structures starting from a set of measurements that are all compatible. We show explicit examples of marginal surgery on a special set of qubit POVMs – which we term *planar symmetric POVMs* – to construct joint measurability structures such as N-cycle and N-Specker scenarios for any integer $N \geq 3$. We also show the realizability of various joint measurability structures with $N \in \{4,5,6\}$ vertices. In particular, for the case of $N = 4$ vertices, we show that *all* possible joint measurability structures are realizable with qubit POVMs. We conjecture that *all* joint measurability structures are realizable with qubit POVMs. This is in contrast to the arbitrarily large dimension required by the construction in R. Kunjwal *et al.*, Phys. Rev. A 89, 052126 (2014). Incidentally, our realization of all N-Specker scenarios on a qubit also renders this previous construction maximally efficient in terms of the dimensions required for realizing arbitary joint measurability structures with POVMs. Along the way, we also prove some new sufficiency conditions for joint measurability of qubit POVMs.

1 Introduction

Measurements in quantum theory exhibit incompatibility, i.e., it is impossible to simulate certain sets of measurements by coarse-graining a single measurement. This lack of joint measurability (or compatibility) is crucial to the demonstration of nonclassicality in quantum theory, e.g., both Bell inequality violations and Kochen-Specker (KS) contextuality are impossible in the absence of incompatible measurements. While the joint measurability (or compatibility) of a set of projective measurements is a binary property, characterized entirely by their pairwise commutativity, the joint measurability of general quantum measurements given by positive operator-valued measures (POVMs) is, in general, not characterized by pairwise commutativity. In this paper we focus on the simplest possible non trivial POVMs - two outcome qubit POVMs termed *binary* qubit POVMs. We review the previously known conditions for their joint measurability and prove new facts and conditions that are summed up in the Section 3

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2 Basic definitions and relations

1. A set of POVMs is jointly measurable or compatible if there is a single (joint) POVM such that by its coarse-graining we obtain the statistics for each of the POVMs from the set.
2. A joint measurability structure is a hypergraph with a set of vertices, V, and a family of (finite) subsets (called hyperedges) of V, denoted $E \subseteq \{e|e \subseteq V\}$. Each vertex denotes a measurement in a set of measurements (indexed by V) and each hyperedge denotes a subset of compatible (or jointly measurable) measurements. Any subset of vertices that do not share a common hyperedge represents an incompatible subset of the given set of measurements.
3. A joint measurability structure on the set of N POVMs of the form $\{\{E_1, E_2\}, \{E_2, E_3\}, \ldots, \{E_N, E_1\}\}$ is termed an N-cycle scenario.
4. A joint measurability structure on the set of N measurements where each of its $N-1$-element subsets is compatible while all N POVMs are incompatible is termed N-Specker's scenario.
5. A special case of both N-cycle and N-Specker's scenario for $N = 3$ is Specker's scenario. This is a joint measurability structure where each pair of three POVMs is compatible while all three of them are incompatible.
6. A binary qubit POVM is a POVM with the two element outcome set $\mathscr{O} = \{\pm 1\}$ defined on a two dimensional complex Hilbert space i.e. this POVM is a mapping $E : \mathscr{P}(\{\pm 1\}) \to \mathscr{B}_+(\mathbb{C}^2)$, where $\mathscr{P}$ is the power set. In the range of a POVM are *effects* - positive semidefinite operators bounded above by identity.
7. Effects of a binary qubit POVM corresponding to outcomes ± 1 have the following representation in Pauli basis: $E(\pm 1) = \frac{1}{2}((1 \pm b)I + \vec{a} \cdot \vec{\sigma})$, where b and $\vec{a} = (a_1, a_2, a_3)$ are respectively the bias and the Bloch vector of POVM E. When $b = 0$ the POVM is termed *unbiased* and the norm of the Bloch vector $\eta = ||\vec{a}|| \in [0, 1]$ is called the *purity* or *sharpness* of the POVM. $\eta = 1$ and $b = 0$ correspond to projective, or *sharp* measurement. When $\eta < 1$ we say that the measurement is noisy where adding noise corresponds to decreasing the purity. Three or more binary qubit POVMs are coplanar if their Bloch vectors are coplanar.

3 Results

In our paper we obtained the following results:

1. We introduced the notion of *geometric equivalence* of qubit POVMs motivated by the fact that certain sets of POVMs that have their effects related by relabelling of outcomes and transforming their Bloch vectors by an isometry of $\mathbb{R}^3$ are physically the same and thus exhibit the same joint measurability structure. We examined how this applies to the set of planar symmetric binary qubit POVMs first introduced in Ref. [2].
2. We introduced a technique of *marginal surgery* - a procedure of tweaking the marginal joint POVM of a subset s_1 of a set s of compatible POVMs to obtain new sets s', of the same cardinality as s, that is incompatible and $s'_1 \subset s'$ of the same cardinality as s_1 that is compatible. Clearly, this technique has the power of generating new joint measurability structures starting from a set of compatible measurements.
3. We applied the marginal surgery to planar symmetric POVMs to obtain the sufficient condition for joint measurability of arbitrary subset of planar symmetric POVMs with the explicit construction of the joint POVM. This condition allows as to construct an N-cycle and an N-Specker's scenario on a qubit with planar symmetric POVMs for all $N \geq 3$.

4. Motivated by the form of the previously mentioned joint POVM we derived the sufficient condition for joint measurability of arbitrary coplanar unbiased binary qubit POVMs with the same puriry with the explicit construction of the joint POVM.
5. Again, motivated by the form of the joint POVM in the previous case we generalize even further - we obtain the sufficient condition for joint measurability of arbitrary binary qubit observables.
6. Applying these and previously known conditions for joint measurability we realize miscellaneous joint measurability structures on a qubit. Particularly, we show that as we add noise to planar symmetric POVMs we progress from N incompatible measurements, passing through N-cycle and N-Specker's scenario and ending up with N compatible measurements. We also pass through other joint measurability structures in between N-cycle and N-Specker's scenario that we cannot completely characterize since we don't have conditions to rule out compatibility - we only have sufficient conditions.
7. We showed that all conceivable joint measurability structures are realizable with unbiased binary qubit observables with $N = 4$ POVMs. All but one of these joint measurability structures are realized as a substructure of a larger structure of $M \geq 4$ planar symmetric POVMs. This one exception is a structure where one observable is pairwise jointly measurable with the other three while these three are pairwise incompatible between themselves.
8. We showed that this structure cannot be realized with coplanar unbiased binary qubit POVMs with the same purity. We provide two examples of how we can overcome this i.e. how we can still realize it on a qubit. In one example we allow for different purities and in the other we allow for non coplanar POVMs.

4 Conclusion and open questions

In this study of joint measurability structures, a number of open questions arise.

1. The sufficient condition for joint measurability of arbitrary subsets of N planar symmetric POVMs is known from previous work to be necessary as well for the cardinality of these subsets of $M = \{2,3,N\}$. The natural question arises: is this condition necessary for arbitrary subset of planar symmetric POVMs. In out paper we make this conjecture in Conjecture 1. The truth of this conjecture would allow us to know the exact joint measurability structure of planar symmetric POVMs for arbitrary purity.
2. The sufficient condition for joint measurability of arbitrary number N of coplanar unbiased binary qubit POVMs with the same purity we derived is known from previous work to be necessary when $N \in \{2,3\}$ so we make a conjecture (Conjecture 2. in our paper) that it is necessary for arbitrary N. The truth of Conjecture 2. implies the truth of Conjecture 1.
3. The sufficient condition for joint measurability of arbitrary number N of binary qubit observables we derived is known from previous work to be necessary as well for $N = 2$ unbiased POVMs and for $N = 3$ coplanar unbiased POVMs where no POVM is a convex combination of other two. Motivate by this we make Conjecture 3. in our paper: the condition we derived is also necessary for arbitrary number N of POVMs with the restrictions that the POVMs are: 1) unbiased; 2) coplanar; 3) no POVM is a convex combination of other two.
4. The question whether all joint measurability structures can be realized on a qubit still remains unanswered. We conjecture that it can (Conjecture 4. in our paper). However, provide the potential counterexample for refuting this Conjecture in the form of $\{\{E_1,E_2\},\{E_1,E_3\},\ldots,\{E_1,E_N\}\}$, $N > 4$ motivated by the difficulties we encountered when realizing this structure for $N = 4$.

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