Contextuality of General Probabilistic Theories

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1 Introduction

Contextuality [3], and more recently, generalized contextuality [6], refer to no-go theorems dismissing context-independent classical models for measured statistics. Besides probabilistic (or *ontological*) models, statistics from operational procedures can also be modeled using the framework of *general probabilistic theories* (GPTs), an example of which is quantum theory.

In this work [5], we prove that any finite-dimensional GPT satisfying the no-restriction hypothesis is ontologically noncontextual if and only if it is simplicial. In the more subtle case of subGPTs, i.e. GPTs for which the no-restriction hypothesis is violated, it is also proven that they are ontologically noncontextual if and only if they are subtheories of simplicial GPTs of the same dimensionality.

2 Ontological Models

Operationally, the *laboratory prescriptions* for preparations and measurements forming the collections $\mathscr{P}{:=}\{\mathsf{P}_k\}$ and $\mathscr{M}{:=}\{\mathsf{M}_j\}$, respectively [6]. The measurement processes with the same number of outcomes are described by a measurable space (Ω,ω), where Ω is the finite set of all outcomes and ω is the σ-algebra of events on it. In ontological model formalism an underlying *ontic* variable space Υ is assumed. Then, given the spaces $\mathscr{Y}$ and $\mathscr{Q}$ of all probability measures on (Υ,υ) (υ is the σ-algebra on Υ) and (Ω,ω), respectively, the ontological model hypothesizes the existence of *convex linear* maps $\mu{:}\mathscr{P}{\to}\mathscr{Y}$ and $\xi{:}\mathscr{M}{\to}\mathscr{Q}$ that assign the *ontic state* μ_{P} to the preparation procedure P and the *ontic measurement* ξ_{M} to the measurement procedure M [6], so that

$$\mu_{\mathsf{P}} : \upsilon \to [0,1], \quad \int_{\Upsilon} d\mu_{\mathsf{P}}(\lambda) = 1, \quad \xi_{\mathsf{M}} : \omega\times\Upsilon \to [0,1], \quad \text{and} \quad \xi_{\mathsf{M}}(\Omega|\lambda) = 1 \quad \forall\lambda \in \Upsilon. \tag{1}$$

The probability of a particular event X in a measurement M given the preparation P is obtained as,

$$p(X|\mathsf{P},\mathsf{M}){=}\int_{\Upsilon} d\mu_{\mathsf{P}}(\lambda)\xi_{\mathsf{M}}(X|\lambda). \tag{2}$$

3 GPTs

A second approach to the abstraction of operational scenarios is known as *general probabilistic theories* (GPTs). Their construction begin with assuming an ordered vector space $\mathscr{V}$ endowed with an inner-product $\langle\cdot,\cdot\rangle$. Each possible event $X{\in}\omega$ is then measured "vector-valuedly" by a *probability vector-valued measure* (PVVM) which is a function $E{:}\omega{\to}\mathscr{V}$ satisfying (i) $E(X){\geqslant}0$ for all $X{\in}\omega$, (ii) $E(\Omega){=}U$

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for a fixed nonzero element $U \in \mathscr{V}$, and (iii) $E(\cup_i X_i) = \sum_i E(X_i)$ for all sequences of disjoint events $X_i \in \omega$. Each vector $E(X_i)$ is called an *effect* where their collection is denoted by $\mathscr{E}$.

It is reasonable [5] to assume that (i) given any operationally legitimate effect $E(X)$, the effect $pE(X)$ for any real number $p \in [0,1]$ is also allowed, and (ii) if E_1 and E_2 are two PVVMs, so is $E_p := pE_1 + (1-p)E_2$ implying that the set of all effects $\mathscr{E}$ is convex. We further assume that $\mathscr{E}$ spans $\mathscr{V}$. These allow us to state a Gleason-type theorem for GPTs as follows [5].

Theorem 1. *Any generalized probability measure $q:\mathscr{E} \to [0,1]$ satisfying (i) $q(E(X)) \geqslant 0$ for all effects $E(X) \in \mathscr{E}$, (ii) $q(U)=1$, and (iii) $q(\sum_i E(X_i)) = \sum_i q(E(X_i))$ for all sequences of effects in $\mathscr{E}$ that satisfy $\sum_i E(X_i) \leqslant U$, must be of the form $q(A) = \langle A, B \rangle$ for all $A \in \mathscr{V}$, for a unique $B \in \mathscr{V}$ which is normalized in the sense that $\langle U, B \rangle = 1$.*

Theorem 1 in conjunction with the *no-restriction hypothesis*, i.e. that given PVVM space $\mathscr{E}$ *all* definable probability measures (q's) on it correspond to physically valid states, delineates the GPT's state space as $\mathscr{S} := \{\rho \in \mathscr{V} | \langle E(X), \rho \rangle \geqslant 0 \; \forall E(X) \in \mathscr{E}, \langle U, \rho \rangle = 1\}$. We denote a GPT by its pair of PVVM and state spaces as $\mathscr{T} := (\mathscr{E}, \mathscr{S})$.

We have shown in the full paper of this abstract [5] that any GPT that does not satisfy the no-restriction hypothesis is obtained as a *subtheory* of possibly (infinitely) many GPTs that do satisfy it by imposing appropriate further constraints, thus named a subGPT.

4 Broad (Non)Contextuality

By the *statistical equivalence* assumption [6, 2], two preparations $\mathsf{P}_1, \mathsf{P}_2 \in \mathscr{P}$ are statistically indiscernible and equivalent, $\mathsf{P}_1 \cong \mathsf{P}_2$, if and only if for every measurement procedure $\mathsf{M} \in \mathscr{M}$ and every event $X \in \omega$ it holds that $p(X|\mathsf{P}_1,\mathsf{M}) = p(X|\mathsf{P}_2,\mathsf{M})$. Similarly, two measurements $\mathsf{M}_1, \mathsf{M}_2 \in \mathscr{M}$ are statistically indiscernible and equivalent, $\mathsf{M}_1 \cong \mathsf{M}_2$, if and only if for every preparation procedure $\mathsf{P} \in \mathscr{P}$ and every event $X \in \omega$ it holds that $p(X|\mathsf{P},\mathsf{M}_1) = p(X|\mathsf{P},\mathsf{M}_2)$. The particular way in which a state or measurement is experimentally realized thus corresponds to an element within an equivalence class and it is called a *context*. The *broad assumption of noncontextuality* of a statistical model for experiments states that our models of physical phenomena should depend only on equivalence classes rather than individual contexts. An ontological model which is noncontextual in the broad sense is called a *noncontextual ontological model (NCOM)* and satisfies $\mathsf{P}_1 \cong \mathsf{P}_2 \Leftrightarrow \mu_{\mathsf{P}_1} = \mu_{\mathsf{P}_2}$ and $\mathsf{M}_1 \cong \mathsf{M}_2 \Leftrightarrow \{\xi_{\mathsf{M}_1}(X|\lambda)\} = \{\xi_{\mathsf{M}_2}(X|\lambda)\}$. Similarly, broad noncontextulaity of GPTs reads as $\mathsf{P}_1 \cong \mathsf{P}_2 \Leftrightarrow \mathsf{P}_1, \mathsf{P}_2 \mapsto \rho$ and $\mathsf{M}_1 \cong \mathsf{M}_2 \Leftrightarrow \mathsf{M}_1, \mathsf{M}_2 \mapsto \{E(X)\}$.

5 Ontological (Non)Contextuality of (Sub)GPTs

We now aks if it is possible to construct ontological models of generic GPTs noting that, with appropriate care, such models will inherit the noncontextuality from the theory leading to NCOMs. In doing so, we replace the preparation and measurement procedures P and M in Eqs. (1) and (2), with their representatives in the theory, $\rho \in \mathscr{S}$ and $\{E(X)\} \subset \mathscr{E}$, respectively. Hence, there should exist *injective* maps $\eta : \mathscr{S} \to \mathscr{Y}$ and $\zeta : \mathscr{E} \to \mathscr{Q}$ that assign the *unique* ontic state η_ρ and ontic measurement ζ_E to each state vector ρ and PVVM E, respectively, such that for all $\lambda \in \Upsilon$ and all events $X \in \omega$, $\eta_\rho(\lambda) \geqslant 0$, $\zeta_E(X|\lambda) \in [0,1]$, and satisfy $\int_\Upsilon d\eta_\rho(\lambda) = 1$, and $\forall \lambda \quad \zeta_E(\Omega|\lambda) = 1$. The probability of a particular event X in a measurement M given the preparation P should then be obtained as

$$p(X|\mathsf{P},\mathsf{M}) = p(X|\rho, E) = \int_\Upsilon d\eta_\rho(\lambda) \zeta_E(X|\lambda). \tag{3}$$

6 Main Results

We now state our first main result regarding the ontological noncontextuality of GPTs.

Theorem 2. *A GPT is ontologically noncontextual if and only if its pure states and nonrefinable sharp effects each form a complete basis for the space $\mathscr{V}$. Equivalently, the GPT must be simplicial meaning that $\mathscr{S}$ and $\overline{\mathrm{conv}\mathscr{E}_{\mathrm{nr}}}$ are simplexes.*

Here we outline the proof of Theorem 2, the complete version of which can be found in [5]. Recall that the maps μ and ξ, and hence η and ζ, are convex linear. Using the fact that $\mathscr{S}$ and $\mathscr{E}$ both span $\mathscr{V}$ and Riesz's theorem we find that η and ζ must be of the forms $\eta_\rho(\lambda)=\langle\rho,F(\lambda)\rangle$ and $\zeta_E(X|\lambda)=\langle E(X),D(\lambda)\rangle$ for $F(\lambda),D(\lambda)\in\mathscr{V}$. Satisfying Eq. (3) then implies that $\mathscr{F}:=\{F(\lambda)\}$ resembles a PVVM whereas $\mathscr{D}:=\{D(\lambda)\}$ is a subset of GPT's state space. After a few more simple steps we find that $\mathscr{F}$ and $\mathscr{D}$ must in fact be the generating sets of closed convex sets $\mathscr{E}$ and $\mathscr{S}$, respectively. We conclude from the latter that the extreme states $\rho\in\mathscr{D}$ and the extreme effects $E(X)\in\mathscr{F}$ must be represented by Dirac measures over the ontic space Υ, that is, (i) $\mathscr{D}\ni\rho\overset{\eta}{\mapsto}\delta_{\lambda_\rho}(\lambda)$ for some $\lambda_\rho\in\Upsilon$ and (ii) $\mathscr{F}\ni E(X)\overset{\zeta}{\mapsto}\delta_{\lambda_{E(X)}}(\lambda)$ for some $\lambda_{E(X)}\in\Upsilon$, where $\delta_a(\beta)$ equals 0 if $a\notin\beta$ and equals 1 if $a\in\beta$ for any measurable subset β. These two enforce that states and effects possess a unique decompositions into nonrefinable extreme elements, hence the theorem.

Theorem 3. *Any subGPT $\mathscr{T}_{\mathrm{sub}}=(\mathscr{E}_{\mathrm{sub}},\mathscr{S}_{\mathrm{sub}})$ over $\mathscr{V}_{\mathrm{sub}}$ admits a NCOM if and only if it is a subtheory of an ontologically noncontextual GPT $\mathscr{T}=(\mathscr{E},\mathscr{S})$ over $\mathscr{V}$ and* $\dim\mathscr{V}_{\mathrm{sub}}=\dim\mathscr{V}=\mathrm{card}\,\Upsilon$.

The proof of Theorem 3 is similar to that of Theorem 2, as detailed in Ref. [5].

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Note added.— Two independent works by Schmid *et al* [4] and Barnum and Lami [1] also present similar results.

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