Hilbert spaces reduced to their orthogonality relation

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A projective Hilbert space together with its usual orthogonality relation is the prototypical example of a so-called orthogonality space: a set equipped with a symmetric and irreflexive binary relation. We present a simple condition that characterises the orthogonality spaces that arise in this way from finite-dimensional inner-product spaces and we observe that the relational structure determines the inner-product space essentially uniquely. On the basis of the resulting correspondence, we moreover discuss structure-preserving maps between orthogonality spaces on the one hand and between projective inner-product spaces on the other hand.

The contribution is based on our papers [9] and [5].

1 Hilbert spaces and their associated orthogonality spaces

Stimulated by the seminal paper of Birkhoff and von Neumann on the "logic of quantum mechanics", intensive efforts have aimed at a reconstruction of the basic model of quantum physics, the Hilbert space, by means of simple and easily interpretable algebraic structures. As an example of what has turned out to be possible, we may mention the characterisation of the infinite-dimensional complex Hilbert space by lattice-theoretic means. As a drawback of the approach, it must often be observed that algebraic structures are of limited use when it comes to describing the mutual relations between, and constructions around, linear spaces.

The present contribution deals with a particular approach along the mentioned lines. Whichever type of total or partial algebra has been chosen in order to grasp the typical properties of a Hilbert space, there is one structural feature that occurs in some form or the other in practically all cases: the orthogonality relation. It was the suggestion of David Foulis and his collaborators to restrict considerations solely to this single binary relation. We arrive at a structure that seems not to be improvable in terms of simplicity. We discuss in this contribution in which sense a Hilbert space can be reduced to its orthogonality relation and explore the relationship between linear maps and those preserving orthogonality.

Definition 1.1. An *orthogonality space* is a non-empty set X equipped with a symmetric, irreflexive binary relation $\perp$, called the *orthogonality relation*.

In the present context, the notion has the first time been systematically studied in [1]. But we note that the definition is not really special; orthogonality spaces are essentially the same as undirected graphs.

The typical example arises, as mentioned, from a Hilbert space. More generally, by a Hermitian space we mean a linear space H over a $\star$-sfield that is equipped with a sesquilinear form $(\cdot,\cdot)$ such that $(x,y) = (y,x)^\star$ for any $x,y \in H$, and $(x,x) = 0$ implies $x = 0$. We define $P(H) = \{[x] \colon x \in H \setminus \{0\}\}$, where $[x]$ is the subspace spanned by $x \in H$, and for $x,y \in H \setminus \{0\}$, we define $[x] \perp [y]$ if $(x,y) = 0$. Then $(P(H),\perp)$ is an example of an orthogonality space.

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Conversely, to get from orthogonality spaces to Hermitian spaces, we consider the following condition [9]. For a subset A of an orthogonality space X, we set $A^\perp = \{e \in X : e \perp a \text{ for all } a \in A\}$.

Definition 1.2. The orthogonality space $(X, \perp)$ is called *linear* if, for any two distinct elements $e, f \in X$, there is a third element g such that $\{e, f\}^\perp = \{e, g\}^\perp$ and exactly one of f and g is orthogonal to e.

By the rank of an orthogonality space, we mean the supremum of the cardinalities of sets of mutually orthogonal elements. We generally assume the rank to be finite.

Theorem 1.3. *Let $(X, \perp)$ be a linear orthogonality space of finite rank $n \geqslant 4$. Then there is a $\star$-sfield K and an n-dimensional Hermitian space H over K such that $(X, \perp)$ is isomorphic to $(P(H), \perp)$.*

With reference to Theorem 1.3, we may add that K is determined up to isomorphism and the inner product is unique up to a scalar multiple.

To identify among the linear orthogonality spaces those that arise from Hilbert spaces is a delicate issue. We have given in [8] conditions characterising finite-dimensional Hermitian spaces over a subfield of $\mathbb{C}$. Based on different ideas, we moreover established a similar conclusion for the case of $\mathbb{R}$ in [10].

2 Categories of orthogonality spaces

We shall elaborate on the correspondence between linear orthogonality spaces on the one hand and Hermitian spaces on the other one. Our concern is to relate the respective structure-preserving maps.

For self-symmetries, the situation is quite transparent. We may combine Piron's and Uhlhorn's versions of Wigner's Theorem [6, 7], to get a correspondence between unitary operators of Hermitian spaces and automorphisms of orthogonality spaces. But in general, orthogonality-preserving maps between orthogonality spaces are unrelated to those preserving linear dependence.

We restrict our focus to orthogonality spaces fulfilling a certain coherence property [5].

Definition 2.1. An orthogonality space $(X, \perp)$ is called *normal* if, for any maximal collection $e_1, \ldots, e_n$ of mutually orthogonal elements and any $1 \leqslant k < n$, if $f \perp e_1, \ldots, e_k$ and $g \perp e_{k+1}, \ldots, e_n$, then $f \perp g$.

With each orthogonality space $(X, \perp)$, we may associate the orthoalgebra $\mathscr{C}(X, \perp) = \{A^{\perp\perp} : A \subseteq X\}$. Normality is equivalent to the requirement that any set of mutually orthogonal elements gives rise, in the expected way, to a Boolean subalgebra of $\mathscr{C}(X, \perp)$. The condition seems natural, since a basic feature of the Hilbert space model of quantum mechanics is the compatibility of orthogonal closed subspaces.

We define the category $\mathscr{NOS}$ of normal orthogonality spaces as follows. A map $\varphi \colon X \to Y$ is a morphism if (1) φ preserves the orthogonality relation and (2) for any maximal collection $e_1, \ldots, e_n$ of mutually orthogonal elements of X and any $f \in Y$, $f \perp \varphi e_1, \ldots \varphi e_n$ implies $f \perp \varphi X$. We may characterise the morphisms of $\mathscr{NOS}$ as transformations that, in a natural sense, map the Boolean subalgebras of $\mathscr{C}(X, \perp)$ into Boolean subalgebras of $\mathscr{C}(Y, \perp)$.

Any linear orthogonality space is normal. We denote the full subcategory of $\mathscr{NOS}$ consisting of the linear orthogonality spaces by $\mathscr{LOS}$.

Let now H and H' be Hermitian spaces (not necessarily over the same $\star$-sfield). We intend to define structure-preserving maps between the associated projective spaces. We surely expect that such a map $\varphi \colon P(H) \to P(H')$ preserves the orthogonality relation. However, to require not more would be unreasonable. Indeed, although linear dependence is a weaker concept than orthogonality, φ could be orthogonality-preserving but so-to-say disregard linear dependency altogether. For morphisms in $\mathscr{LOS}$, however, we do get a reasonable correspondence and the relevant concept is the following [4, 2]: φ is a lineation if it preserves the triple relation of being on a line. This means that whenever $[x], [y], [z]$ are contained in a two-dimensional subspace of H, then $\varphi[x], \varphi[y], \varphi[z]$ are contained in a two-dimensional subspace of H'.

Theorem 2.2. *Let H and H' be finite-dimensional Hermitian spaces. Then a map $\varphi\colon P(H) \to P(H')$ is a morphism in $\mathscr{LOS}$ if and only if φ is an orthogonality-preserving lineation.*

The remainder of this work is devoted to the question whether orthogonality-preserving lineations are induced by maps between the underlying spaces.

By a theorem of Machala [4], a non-degenerate lineation between projective spaces is induced by a generalised semilinear map. Here, a lineation is called non-degenerate if (L1) its image is not contained in a two-dimensional subspace and (L2) the image of any line is never two-element.

Lemma 2.3. *Let H and H' be Hermitian spaces of finite dimension $\geqslant 3$ and let $\varphi\colon P(H) \to P(H')$ be a morphism in $\mathscr{LOS}$. Then φ fulfils* (L1)*. Moreover, if φ does not fulfil* (L2)*, then there exists a two-valued measure on a* 3*-dimensional subspace of H.*

Thus the question about the representability of morphisms in $\mathscr{LOS}$ is reduced to the question of the existence of two-valued measures. Adapting Piron's proof of Gleason's Theorem [6], we can make a statement in case of a particular type of $\star$-sfield. We recall that an ordered field is called *Euclidean* if any positive element is a square.

Theorem 2.4. *A three-dimensional positive definite Hermitian space over a Euclidean subfield of the reals does not possess two-valued measures.*

Finally, we should wonder whether an orthogonality-preserving lineation is induced by a generalised semiunitary map, that is, a generalised semilinear map preserving also the inner product. In case of the particular assumption of Theorem 2.4, the answer is affirmative.

Theorem 2.5. *Let H and H' be positive definite Hermitian spaces of finite dimension $\geqslant 3$ over Euclidean subfields of the reals. Then any morphism in $\mathscr{LOS}$ between $P(H)$ and $P(H')$ is induced by a generalised semiunitary map.*

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