

# The ZX& calculus: A complete graphical calculus for classical circuits using spiders

Cole Comfort

Department of Computer Science, University of Oxford

We give a complete presentation for the fragment, ZX&, of the ZX-calculus generated by the Z and X spiders (corresponding to copying and addition) along with the not gate and the and gate. To prove completeness, we freely add units and counits to the category TOF generated by the Toffoli gate and ancillary bits, showing that this yields the strictification of spans of powers of the two element set; and then perform a two way translation between this category and ZX&. A translation to some extension of TOF, as opposed to some fragment of the ZX calculus, is a natural choice because of the multiplicative nature of the Toffoli gate. To this end, we show that freely adding counits to the semi-Frobenius algebra of a discrete inverse category is the same as computing the “environment structure” of the classical structures of the base discrete inverse category. We show that in this setting, the classical channels and the discrete Cartesian completion are the same constructions. Therefore, in the case of TOF, freely adding a counit, constructing the category of quantum channels, and computing the discrete Cartesian completion are all equivalent to partial functions between powers of the two element set. By glueing together the free counit completion and the free unit completion, this yields qubit multirelations.

## 1 Introduction

In this paper a complete set of identities is provided for the fragment, ZX&, of the ZX-calculus, generated by black and white spiders, the not gate and the and gate. We show that this is a universal and complete presentation of “qubit multirelations,” or equivalently  $2^n \times 2^m$  dimensional matrices over  $\mathbb{N}$ . To prove completeness and universality requires much exposition. Along the way we show that the category of classical channels of a discrete inverse category is the discrete Cartesian completion of that discrete inverse category. We then show that the corresponding environment structure is precisely the free counit completion of the chosen Frobenius structure. This allows us to present the discrete Cartesian completion of, TOF, the category generated by the Toffoli gate,  $|1\rangle$  and  $\langle 1|$  by only adding the  $|+\rangle$  state and the unitality equation. By freely adding both the unit and counit to TOF, corresponding to unnormalized  $|+\rangle$  and  $\langle +|$ , this yields an isomorphism with spans between ordinals  $[2^n]$ ,  $n \in \mathbb{N}$ .

The identities which are given by this two way translation are almost exactly the union of the complete identities for Boolean functions [26, Thm. 10] (functions of type  $\mathbb{Z}_2^n \rightarrow \mathbb{Z}_2$ ) and the identities

for interacting Hopf algebras over  $\mathbb{Z}_2$  [3, Def. 5.1]; however, we don't require that  $\langle +|+ \rangle = 1$ , so some axioms are appropriately scaled. These classes of circuits, and these identities for that matter, are nothing new; however, we provide a completeness result, as well as a structural account of how the full classical qubit fragment of FHilb can be obtained from adding discarding and codiscarding to the full classically reversible Boolean fragment. In fact, some of these identities are presented in [22, Chap. 5], and they are used in the ZH-calculus [1, 33], as well as in some presentations of the ZX-calculus with the triangle generator as a primitive [27, 32]. This is particularly unsurprising for the latter, [32], where the author proves completeness of the ZX-calculus over arbitrary semirings, which subsumes the completeness result herein. Albeit, the presentation given here is substantially simpler. It should be said, that ZX& is not a ZX\* calculus in the sense of [7], because the and gate is not a spider. ZX& should be instead thought of as the “classical fragment” of the phase-free ZH-calculus: retaining the monoid for “and” without  $H$ -boxes. From this presentation only natural-number  $H$ -boxes can be derived.

We assume familiarity with the theory of monoidal categories and categorical quantum mechanics. Most of the paper will be devoted to reviewing the required categorical machinery of restriction and inverse categories, and developing it further, in order to prove the main result. With all of mathematics reviewed and developed in generality, the desired result follows from abstract nonsense after a mechanical calculation.

In Section 2, the theory of restriction categories and inverse categories is reviewed. In Section 3, we construct classical channels in the setting of discrete inverse categories, showing that the “environment structures” of the classical channels corresponds to adding a counit to the base discrete inverse category. Finally, in Section 4, we actually compute the (co)unit completion of TOF. We show that this category has a much more canonical presentation, ZX&, in terms of interacting monoids/comonoids which very much resembles the ZH calculus. We also show that this category is isomorphic to the category spans between ordinals  $[2^n]$ ,  $n \in \mathbb{N}$ .

## 2 Restriction and Inverse Categories

Restriction and inverse categories provide a categorical semantics for partial computing and reversible computing, respectively. We review how weakened products can be constructed in both settings; relating one to the other.

**Definition 2.1.** [10, §2.1.1] *A restriction category is a category along with a restriction operator:*

$$(A \xrightarrow{f} B) \mapsto (A \xrightarrow{\bar{f}} A)$$

*such that:*<sup>1</sup>

---

<sup>1</sup>Using diagrammatic composition.

$$[\mathbf{R.1}] \quad \bar{f}f = f \qquad [\mathbf{R.2}] \quad \bar{f}\bar{g} = \bar{g}\bar{f} \qquad [\mathbf{R.3}] \quad \bar{f}\bar{g} = \overline{fg} \qquad [\mathbf{R.4}] \quad f\bar{g} = \overline{fg}f$$

Maps of the form  $\bar{f}$  are called *restriction idempotents*. The canonical example of a restriction category is  $\mathbf{Par}$ , sets and partial maps. The restriction in this case, just restricts partial functions to their domain of definition.

Restriction categories have a partial order on homsets given by  $f \leq g \iff \bar{f}g = f$ .

A map  $f$  in a restriction category is called a **partial isomorphism**, in case there exists a map  $g$  called the *partial inverse* of  $f$  so that  $fg = \bar{f}$  and  $gf = \bar{g}$ . Similarly, a map  $f$  in a restriction category is **total** if  $\bar{f} = 1$ . Denote the subcategories of partial isomorphisms and total maps of a restriction category  $\mathbb{X}$ , respectively by  $\mathbf{ParIso}(\mathbb{X})$  and  $\mathbf{Total}(\mathbb{X})$ .

**Example 2.2.** [29, p. 101] [11, §5] A **counital copy category** (or a *p-category* with a one element object) is a monoidal category with a family of commutative comonoids on every object compatible with the monoidal structure, with a natural comultiplication. This gives a restriction via copying and then discarding:

$$\boxed{\bar{f}} := \text{---} \bigcirc \begin{array}{l} \nearrow \boxed{f} \text{---} \bigcirc \\ \searrow \text{---} \end{array}$$

**Definition 2.3.** [10, §3.1] A **stable system of monics**  $\mathcal{M}$  of  $\mathbb{X}$  is a collection of monics in  $\mathbb{X}$  containing all isomorphisms; where for any cospan  $X \xrightarrow{f} Z \xleftarrow{m} Y$  in  $\mathbb{X}$ , where  $m'$  is in  $\mathcal{M}$ , the following pullback exists:

$$\begin{array}{ccc} & W & \\ m' \swarrow & & \searrow f' \\ X & & Y \\ f \searrow & Z & \swarrow m \end{array}$$

Where  $m'$  is in  $\mathcal{M}$ .

Stable systems of monics allow one to represent the domains of definition of a partial functions as a subobjects:

**Definition 2.4.** [10, §3.1] Given a stable system of monics  $\mathcal{M}$  in a category  $\mathbb{X}$ , the **partial map category**  $\mathbf{Par}(\mathbb{X}, \mathcal{M})$  is given by the same objects as in  $\mathbb{X}$  where morphisms  $X \rightarrow Y$ , given by isomorphism classes of spans  $X \xleftarrow{m} Z \xrightarrow{f} Y$  where  $f$  is a map in  $\mathbb{X}$  and  $m$  is a map in  $\mathcal{M}$ . Composition is given by pullback and the identity is given by the trivial span.

Partial map categories have a restriction structure given by:  $(X \xleftarrow{m} Z \xrightarrow{f} Y) \mapsto (X \xleftarrow{m} Z \xrightarrow{m} X)$ . Moreover, a partial isomorphism is a span  $X \xleftarrow{e} Z \xrightarrow{m} Y$  where  $e, m \in \mathcal{M}$ ; the partial inverse given by  $Y \xleftarrow{m} Z \xrightarrow{e} X$ .

$\mathbf{Par}$  is equivalently the partial map category  $\mathbf{Par}(\mathbf{Set}, \mathcal{M})$  where  $\mathcal{M}$  is all monics in  $\mathbf{Set}$ .

Let  $\text{Span}^\sim(\mathbb{X})$  denote the category given by isomorphism classes of spans over  $\mathbb{X}$ . Given a stable system of monics  $\mathcal{M}$  over  $\mathbb{X}$ , if  $\mathbb{X}$  is finitely complete, then  $\text{Span}^\sim(\mathbb{X})$  exists, and thus, there is a faithful functor  $\text{Par}(\mathbb{X}, \mathcal{M}) \rightarrow \text{Span}^\sim(\mathbb{X})$ .

**Definition 2.5.** [10, §2.3.2] An **inverse category** is a restriction category in which all maps are partial isomorphisms. The subcategory of partial isomorphisms of  $\text{Par}$  is called  $\text{Pinj}$ .

Inverse categories can be presented with a dagger functor taking maps to their partial inverses:

**Theorem 2.6.** [10, Thm. 2.20] A restriction category  $\mathbb{X}$  is an inverse category if and only if there is a dagger functor  $(-)^\circ : \mathbb{X}^{\text{op}} \rightarrow \mathbb{X}$  such that for all  $X \xleftarrow{f} Z \xrightarrow{g} Y$ :

$$ff^\circ f = f \quad ff^\circ gg^\circ = gg^\circ ff^\circ$$

Since restriction categories and inverse categories give a categorical semantics for partial computing and reversible computing, respectively, it is natural to ask when these categories have copying.

In the case of restriction categories, one must weaken the notion of the product to lax products using the partial order enrichment:

**Definition 2.7.** [11] A restriction category has **binary restriction products**, when for all objects  $X, Y$ , there exists an object  $X \times Y$  and total maps  $X \xleftarrow{\pi_0} X \times Y \xrightarrow{\pi_1} Y$ , so that for all objects  $Z$  and all maps  $X \xleftarrow{f} Z \xrightarrow{g} Y$ , the following diagram commutes there exists a unique  $Z \xrightarrow{\langle f, g \rangle} X \times Y$  making the diagram commute:

$$\begin{array}{ccccc} & & Z & & \\ & f \swarrow & & \searrow g & \\ X & \xleftarrow{\pi_0} & X \times Y & \xrightarrow{\pi_1} & Y \\ & \uparrow \geq & \downarrow \leq & & \\ & \langle f, g \rangle & & & \end{array}$$

so that  $\overline{\langle f, g \rangle} \pi_0 f = \langle f, g \rangle \pi_0$  and  $\overline{\langle f, g \rangle} \pi_1 g = \langle f, g \rangle \pi_1$ ; where additionally  $\overline{\langle f, g \rangle} = \overline{f} \overline{g}$ .

A restriction category has a **restriction terminal object**  $\top$  when for all objects  $X$ , there exists a unique total map  $!_X : X \rightarrow \top$  such that  $f!_Y = \overline{f}!_X$ .

A restriction category with a restriction terminal object and binary restriction products is a **Cartesian restriction category**.

An object  $A$  in a restriction category with restriction products is **discrete** when the diagonal map  $\Delta_X := \langle 1_X, 1_X \rangle$  is a partial isomorphism. A restriction category is discrete when all objects are discrete. Discrete Cartesian restriction categories are said to have restriction products.

**Theorem 2.8.** [11, Thm. 5.2] The structure of a counital copy category structure is precisely that of a Cartesian restriction category.

**Proposition 2.9.** [11, §5.1]

If  $\mathbb{X}$  is a discrete Cartesian restriction category, then  $\text{Total}(\mathbb{X})$  is Cartesian.

$\text{Par}$  is a canonical example of a discrete Cartesian restriction category; the restriction product is given by the Cartesian product on underlying sets and the terminal object is the singleton set.

The weakened notion of products in restriction categories is not satisfying for inverse categories because it does not impose enough equations governing the interaction between the diagonal map and its partial inverse.

**Definition 2.10.** [21, Def. 4.3.1] A symmetric monoidal inverse category  $\mathbb{X}$  is a **discrete inverse category** when there is a natural, special commutative  $\dagger$ -semi-Frobenius algebra<sup>2</sup> on every object (where the (co)multiplications are drawn as white bubbles) compatible with the tensor product:

Where the tensor product is also required to preserve restriction in both components.

In a discrete inverse category, restriction idempotents are prephases for the Frobenius algebra, so that:

Discrete inverse categories are the “right” notion of weakened products for monoidal inverse categories:

**Theorem 2.11.** [21, Thm. 5.2.6] There is an equivalence of categories between the category of discrete inverse categories and the category of discrete Cartesian categories.

To go from discrete Cartesian restriction categories to discrete inverse categories, one takes the subcategory of partial isomorphisms. The other direction is less trivial; in particular, this involves adding a restriction terminal object via the following construction which “adds a history” to a partial isomorphism:

**Definition 2.12.** [21, Def. 5.1.1] Given a discrete inverse category  $\mathbb{X}$ , define its **discrete Cartesian completion**  $\tilde{\mathbb{X}}$  as the category with:

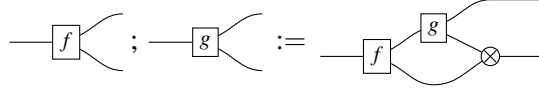
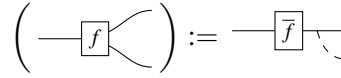
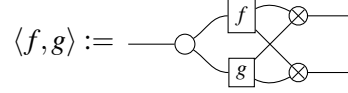
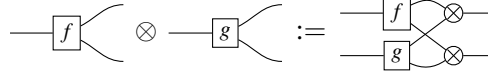
**Objects:** The same objects as  $\mathbb{X}$ .

**Maps:**

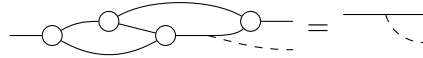
$$\frac{X \xrightarrow{f} Y \otimes S \in \mathbb{X}}{X \xrightarrow{(f,S)} Y \in \tilde{\mathbb{X}}}$$

Where two parallel maps  $X \xrightarrow{(f,S),(g,T)} Y$  are equivalent when either (both conditions are equivalent):

<sup>2</sup>The “semi” adjective on Frobenius just means that the a semigroup and cosemigroup are interacting instead of a monoid and comonoid.

**Composition:****Identity:****Restriction:****Restriction product:****Restriction terminal map:****Tensor product:****Tensor unit:** *The same as in  $\mathbb{X}$ .***Example 2.13.** [21, Ex. 5.3.3]  $\widetilde{\text{Pinj}}$  is Par.*Proof.* For a partial function  $f : X \rightarrow Y$ ,  $\{(x, (y, x)) \mid (x, y) \in f\} / \sim$  is a partial isomorphism.  $\square$ **Lemma 2.14.** *The canonical functor  $\iota : \mathbb{X} \rightarrow \widetilde{\mathbb{X}}$  is faithful.*

The proof is contained in §A.

**Lemma 2.15.** *The induced Frobenius algebra structure in  $\widetilde{\mathbb{X}}$  is counital.**Proof.* For all  $X$ , the map  $X \rightarrow (X \otimes X) \otimes I$  in  $\widetilde{\mathbb{X}}$  induced by the Frobenius algebra in  $\mathbb{X}$  has a counit given by the unitor  $X \rightarrow I \otimes X$  since, in  $\mathbb{X}$ : $\square$ 

### 3 Categorical quantum mechanics and completely positive maps

The CPM construction gives a notion of quantum channels for any  $\dagger$ -compact closed category [31]. The  $\dagger$ -Frobenius algebras in the base category induce idempotents in CPM corresponding to decohering quantum channels. By splitting these idempotents one obtains the  $\text{CP}^*$  construction of [13]: yielding classical channels between finite dimensional  $C^*$ -algebras when applied to  $\text{FHilb}$ . However, both the CPM construction and the  $\text{CP}^*$  construction can not be applied to  $\text{Hilb}$  in general because unlike  $\text{FHilb}$ , it is not compact closed. The  $\text{CP}^\infty$  construction [12] generalizes the

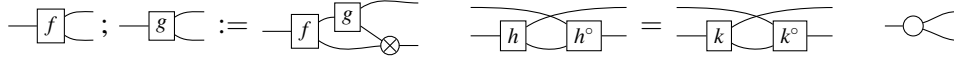


Figure 1: Composition of representatives  $f;g$ ; equivalence relation  $h \sim k$ ; decoherence map in  $\text{CP}^\infty$ .

CPM construction to (non compact closed)  $\dagger$ -symmetric monoidal categories, by unbending the cups/caps and, identifying two Kraus super-maps when they act the same on all positive test maps.

To generalize the  $\text{CP}^*$  construction to  $\dagger$ -semi-Frobenius algebras, one must combine the  $\text{CP}^*$  and  $\text{CP}^\infty$  constructions, as the compact closed structure is no longer taken for granted. We show that splitting the idempotent in the  $\text{CP}^\infty$  construction induced by the chosen semi-Frobenius algebra for every object in a discrete inverse category is precisely its discrete Cartesian completion.<sup>3</sup> The following Lemma is needed to prove this fact:

**Lemma 3.1.** *Given two parallel maps  $X \xrightarrow{f,g} Y \otimes Z$  in a discrete inverse category:*

$$-[f] = -[g] \iff \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \text{---} [f] = \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \text{---} [g]$$

*Proof.* Clearly if  $f = g$ , then the right hand side of the equation holds. Suppose that the right hand side equation holds. Then:

$$\begin{aligned} -[f] &= -[f] \text{ } [f^\circ] = \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \text{---} [f] = \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \text{---} [f] = \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \text{---} [f] = \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \text{---} [f] \\ &= \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \text{---} [g] = \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \text{---} [g] = \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \text{---} [g] = -[g] \end{aligned}$$

□

**Lemma 3.2.** *Given two maps  $X \xrightarrow{f} Y \otimes S$  and  $X \xrightarrow{g} Y \otimes T$ , in a discrete inverse category:*

$$\begin{aligned} -[g] \text{ } [g^\circ] \text{ } [f] &= -[f] \iff \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \text{---} [f] = \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \text{---} [f] = \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \text{---} [f] \\ &\iff \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \text{---} [f] = \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \text{---} [f] = \begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \end{array} \text{---} [f] \end{aligned}$$

<sup>3</sup>Although, composition of the  $\text{CP}^\infty$  construction applied to a discrete inverse category is not obviously well-defined unless the base category embeds in a compact closed category

*Proof.* First note:

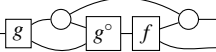
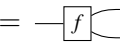
$$\begin{aligned}
 & \text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3} = \text{Diagram 4} \\
 & = \text{Diagram 5} = \text{Diagram 6}
 \end{aligned}$$

So that we only have to prove the first biconditional. Suppose that the left hand side holds, then:

$$\begin{aligned}
 & \text{Diagram 1} = \text{Diagram 2} \\
 & = \text{Diagram 3} = \text{Diagram 4} \\
 & = \text{Diagram 5} = \text{Diagram 6} \\
 & = \text{Diagram 7} = \text{Diagram 8}
 \end{aligned}$$

Conversely, suppose that the right hand side holds. Then:

$$\begin{aligned}
 & \text{Diagram 1} = \text{Diagram 2} = \text{Diagram 3} \\
 & = \text{Diagram 4} = \text{Diagram 5} \\
 & = \text{Diagram 6}
 \end{aligned}$$

Thus, by Lemma 3.1 :  = 

□



The natural question arises: can we characterize classical channels in this setting, algebraically in terms of a discarding morphism, without performing any doubling. In other words, is there some notion of “environment structure” [16] for the *classical* channels of discrete inverse categories:

**Definition 3.3.** *Given a discrete inverse category  $\mathbb{X}$ , define the counital completion of  $\mathbb{X}$ ,  $c(\mathbb{X})$  to have the same objects and maps of  $\mathbb{X}$ , except with a freely adjoined counit  $!_X : X \rightarrow I$  to the chosen semi-Frobenius algebra on  $X$ , for each object in  $\mathbb{X}$  compatible with the monoidal structure.*

**Lemma 3.4.**  *$c(\mathbb{X})$  is a discrete Cartesian restriction category.*

*Proof.* This is clearly a counital copy category, with a restriction terminal object given by the tensor unit. Moreover, because the Frobenius structure is special, it is also discrete.  $\square$

**Lemma 3.5.** *Given a discrete inverse category  $\mathbb{X}$ ,  $c(\mathbb{X})$  and  $\widetilde{\mathbb{X}}$  are isomorphic as discrete Cartesian restriction categories.*

The proof is contained in §B.

## 4 ZX&

In this section, we add a unit and counit to the Frobenius algebra in TOF by glueing its counital completion and unital completion together. We then give a presentation of this category in terms of the self-dual compact closed prop ZX& generated by the copy and addition spiders, the not gate and the and gate via a two-way translation.

**Definition 4.1.** [8] *The category TOF is the prop generated by the Toffoli gate and ancillary bits, satisfying the equations in §D Figure 5.*

**Theorem 4.2.** [8] *TOF is isomorphic to partial isomorphisms between ordinals  $[2^n]$ ,  $n \in \mathbb{N}$ .*

What do discrete inverse categories with units and counits freely adjoined look like? We can compute this for categories such as Pinj and TOF:

**Lemma 4.3.** *Consider a finitely complete category  $\mathbb{X}$ , and a stable system of monics  $\mathcal{M}$ , so that  $\text{Span}^\sim(\mathcal{M})$  is a discrete inverse category with discrete Cartesian completion  $\text{Par}(\mathbb{X}, \mathcal{M})$ . Then adding a unit and counit to the chosen semi-Frobenius structure in  $\text{Span}^\sim(\mathcal{M})$  yields  $\text{Span}^\sim(\mathbb{X})$ .*

The proof is contained in §C.

If  $f$  is a partial isomorphism between finite sets, then the white spiders correspond to the classical structure for the chosen computational basis. For the interpretation into FHilb via the  $\ell_2$  functor, this means that in the qubit case, the unit and counit correspond to the unnormalized  $|+\rangle$  and  $\langle +|$  state and effect. Call the category TOF with the unit and counit adjoined,  $\widehat{\text{TOF}}$ :

**Corollary 4.4.**  *$\widehat{\text{TOF}}$  is isomorphic to spans between ordinals  $[2^n]$  for  $n \in \mathbb{N}$ .*

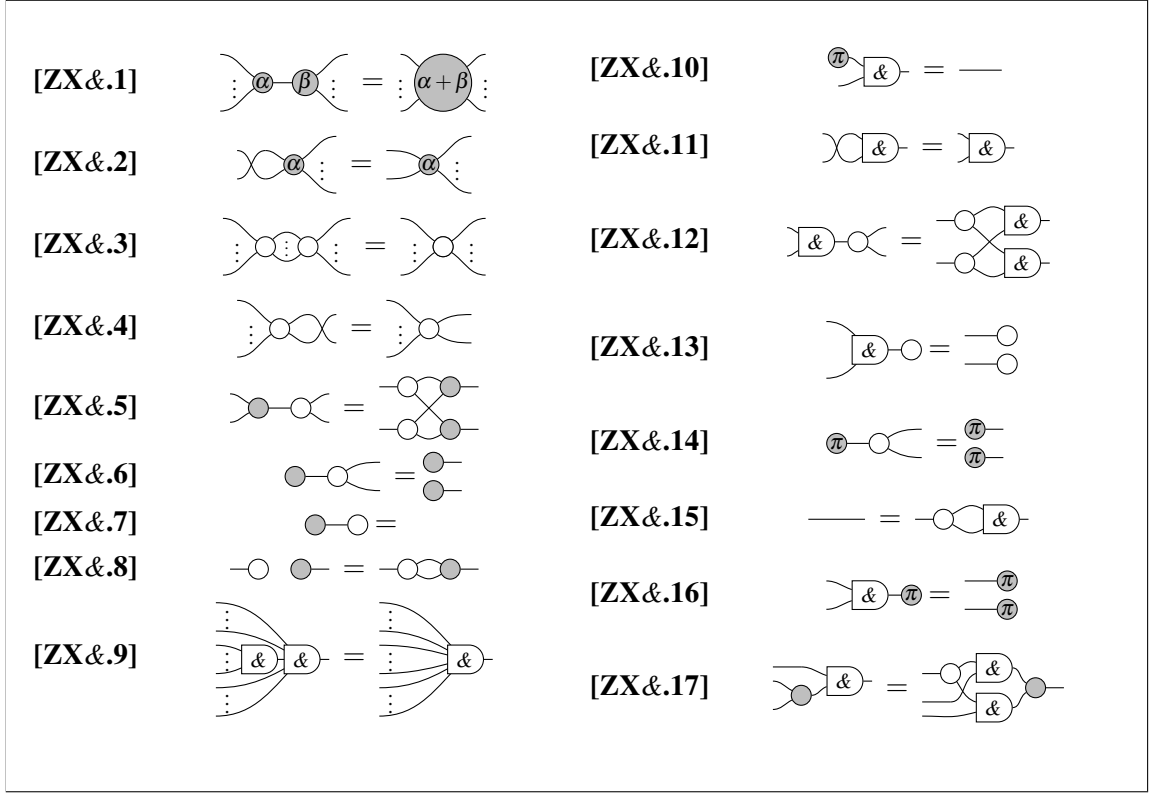


Figure 2: The identities of ZX&, where  $\alpha, \beta \in \{0, \pi\}$  and a blank grey spider has angle 0.

We give a more elegant presentation of this category in terms of interacting monoids and comonoids:

**Definition 4.5.** Consider the self dual prop ZX& generated by the addition spider with phases in  $\{0, \pi\}$ , the copy spider and the monoid for conjunction satisfying the identities given in Figure 2.

One can interpret the generators as logical connectives and open wires as variables, similar to the regular logic [5], or the logic of a Cartesian bicategory [6], except we forget the 2-cells in ZX&. The decorated black spiders correspond to fixed variables and xor. White (co)multiplications (co)copy variables; the white unit is existential quantification and the counit is discarding. The relations are open  $\Sigma_1$  Boolean formulas augmented with copying and discarding as well as duals; the open variables correspond to distinguished inputs and outputs.

The identities of ZX& can also be interpreted by freely taking the coproduct of the free prop of commutative (co)monoids  $\dagger\text{-PROP}$   $3 \times 2$  times, modulo various (undirected) distributive laws, and monoid maps. The distributive laws are summarized in Figure 3 (the duals under diagonal are omitted). The spider rules implicitly identify the (co)units of the  $\dagger$ -compact closed structure induced

by  $Z$  and  $X$ ; which is needed for completeness.

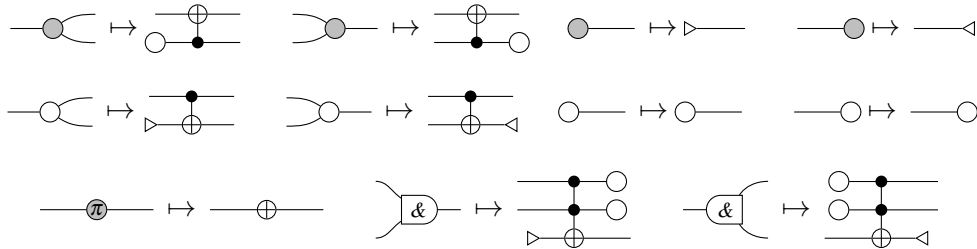
$\lambda$	$Z$	$X$	$\&$	$Z^\dagger$	$X^\dagger$	$\&^\dagger$
$Z$	Comm. monoid			Extraspecial commutative $\dagger$ -Frobenius algebra	Hopf algebra $^\ddagger$ with $s=1$	Special bialgebra
$X$		Comm. monoid		Hopf algebra $^\ddagger$ with $s=1$	Commutative $\dagger$ -Frobenius algebra	
$\&$			Comm. monoid	Special bialgebra		
$Z^\dagger$				Cocomm. comonoid		
$X^\dagger$					Cocomm. comonoid	
$\&^\dagger$						Cocomm. comonoid

Figure 3: Generating distributive laws of  $ZX\&$ .  $\ddagger$ : The unit  $Z$  of is only copied by the comultiplication of  $X^\dagger$  by a factor of 2.

Additionally, [ZX&.16] states that the counit of  $\&^\dagger$  is copied by  $\&$ ; ie. the counit is a monad map from  $\&$  to the trivial monad. [ZX&.17], alongside the other axioms, using the compact structure, is asserting that  $\widehat{\&}$  is a monad map from  $Z$  to  $Z \otimes X$ :

$$\widehat{\&} := \text{---} \boxed{\&} \text{---} \quad \text{where} \quad \text{---} \text{---} \boxed{\&} \text{---} \stackrel{[\text{ZX}\&.\text{17}]}{=} \text{---} \boxed{\&} \text{---} \text{---} \text{---} \boxed{\&} \text{---} \text{---} \text{---} \stackrel{\text{Lem.E.7}}{=} \text{---} \text{---} \text{---} \text{---}$$

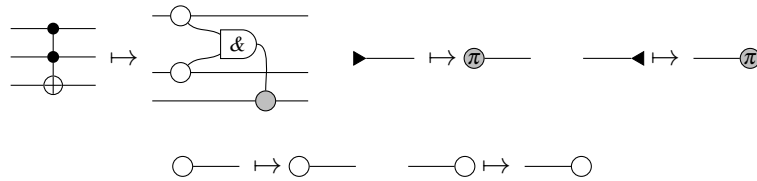
**Proposition 4.6.** Consider the interpretation  $\llbracket - \rrbracket_{ZX\&} : ZX\& \rightarrow \widehat{\text{TOF}}$  taking:



This interpretation is a strict symmetric  $\dagger$ -monoidal functor.

See §E.3 for the proof.

**Proposition 4.7.** Consider the interpretation  $\llbracket - \rrbracket_{\widehat{\text{TOF}}} : \widehat{\text{TOF}} \rightarrow ZX\&$  taking:



This interpretation is a strict symmetric  $\dagger$ -monoidal functor.

See §E.4 for the proof.

**Theorem 4.8.** *The interpretation functors  $\llbracket - \rrbracket_{\text{ZX\&}}$  and  $\llbracket - \rrbracket_{\widehat{\text{TOF}}}$  are inverses, so that  $\widehat{\text{TOF}}$  and ZX& are isomorphic as strongly compact closed props.*

See §E.5 for the proof.

Recall the following proposition:

**Proposition 4.9.** *[4, Prop. 2.6]<sup>4</sup> The category  $\text{Span}^{\sim}(\text{FinOrd})$  equipped with the Cartesian product is monoidally equivalent to the category of (finite) matrices over the natural numbers and the Kronecker product.*

Thus,

**Corollary 4.10.** *ZX& is complete for the prop of  $2^n \rightarrow 2^m$  matrices over the natural numbers.*

## 5 Conclusion

There are various other directions which could be pursued. One could also ask if there is a normal form for ZX& induced by the presentation in terms of distributive laws and monoid maps, using the correspondence between strict factorization systems and distributive laws in spans [30]. It would also be interesting to investigate the 2-categorical structure of ZX&; presenting the corresponding category of relations as a Frobenius theory [2] using the partial order enrichment of TOF.

Another immediate direction would be to add the white  $\pi$  phase to ZX& to obtain an approximately universal graphical calculus for quantum computing using only distributive laws and monoid maps. In such a fragment, one could construct the and gate for the  $X$  basis; perhaps expanding the table of distributive laws in Figure 3 to be complete for an approximately universal fragment of quantum computing, furthering the general programme of [3, 20] decomposing circuits using distributive laws. This approach is contrasted to considering H-boxes as primitives, as in the phase-free fragment of the ZH-calculus [33]—in ZX&+the white  $\pi$  phase, the unnormalized Hadamard gate is derived. Perhaps proving the minimality of the axioms using this presentation might be easier, although we do not prove minimality in this paper.

It would also be interesting to investigate the connection to the ZH-calculus and triangle fragments of the ZX-calculus; in particular, in regard to natural number labelled H-boxes, as in [19]. These gates can be represented in string diagrams. The diagram of the triangle can be interpreted as the assertion  $x \wedge \neg y = \perp$  which is equivalent to the material implication  $x \Rightarrow y$ .

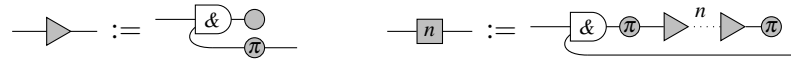


Figure 4: Triangles and H-boxes in ZX&, for  $n \in \mathbb{N}$ .

<sup>4</sup>In [4], they do not prove this equivalence is monoidal, but it is an obvious corollary.

## Acknowledgements

I would like to thank Sivert Aasnæss, Niel de Beaudrap, Robin Cockett, Lukas Heidemann, Aleks Kissinger, JS Lemay and James Hefford, Konstantinos Meichanetzidis and John van de Wetering for useful discussions. Especially Robin for pointing out that the conditions for the  $CP^*$  construction and the discrete Cartesian completion were equivalent, *in general*; Aleks for seeing the decomposition of the triangle; Niel for asking about the relation to natural number labeled H boxes; and John for private discussions involving the simplification of the axioms of the phase free ZH-calculus.

I also gratefully acknowledge support from the Clarendon fund.

## References

- [1] M Backens & A Kissinger (2019): *ZH: A Complete Graphical Calculus for Quantum Computations Involving Classical Non-linearity*. *Electronic Proceedings in Theoretical Computer Science* 287, pp. 23–42.
- [2] Filippo Bonchi, Dusko Pavlovic & Pawel Sobocinski (2017): *Functorial semantics for relational theories*. *arXiv preprint arXiv:1711.08699*.
- [3] Filippo Bonchi, Paweł Sobociński & Fabio Zanasi (2017): *Interacting Hopf algebras*. *Journal of Pure and Applied Algebra* 221(1), pp. 144–184.
- [4] Roberto Bruni & Fabio Gadducci (2003): *Some algebraic laws for spans (and their connections with multirelations)*. *Electronic Notes in Theoretical Computer Science* 44(3), pp. 175–193.
- [5] Carsten Butz (1998): *Regular categories and regular logic*. *BRICS Lecture Series*.
- [6] Aurelio Carboni & Robert FC Walters (1987): *Cartesian bicategories I*. *Journal of pure and applied algebra* 49(1-2), pp. 11–32.
- [7] Titouan Carrette & Emmanuel Jeandel (2020): *A recipe for quantum graphical languages*.
- [8] JRB Cockett & Cole Comfort (2018): *The category TOF*. *arXiv preprint arXiv:1804.10360*.
- [9] JRB Cockett, Cole Comfort & Priyaa Srinivasan (2017): *The category CNOT*. *arXiv preprint arXiv:1707.02348*.
- [10] JRB Cockett & Stephen Lack (2002): *Restriction categories I: categories of partial maps*. *Theoretical computer science* 270(1-2), pp. 223–259.
- [11] JRB Cockett & Stephen Lack (2007): *Restriction categories III: colimits, partial limits and extensivity*. *Mathematical Structures in Computer Science* 17(4), pp. 775–817.
- [12] Bob Coecke & Chris Heunen (2016): *Pictures of complete positivity in arbitrary dimension*. *Information and Computation* 250, pp. 50–58.
- [13] Bob Coecke, Chris Heunen & Aleks Kissinger (2016): *Categories of quantum and classical channels*. *Quantum Information Processing* 15(12), pp. 5179–5209.
- [14] Bob Coecke & Aleks Kissinger (2017): *Picturing quantum processes*. Cambridge University Press.
- [15] Bob Coecke, Eric Oliver Paquette & Dusko Pavlovic (2009): *Classical and quantum structuralism. Semantic Techniques in Quantum Computation*, pp. 29–69.

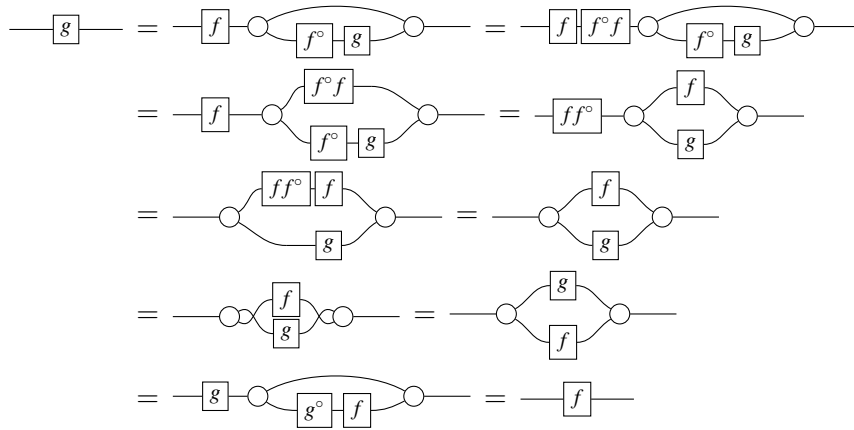
- [16] Bob Coecke & Simon Perdrix (2010): *Environment and classical channels in categorical quantum mechanics*. In: *International Workshop on Computer Science Logic*, Springer, pp. 230–244.
- [17] Bob Coecke, John Selby & Sean Tull (2017): *Two roads to classicality*. *arXiv preprint arXiv:1701.07400*.
- [18] Cole Comfort (2019): *Classifying Reversible Logic Gates with Ancillary Bits*. Master’s thesis, University of Calgary.
- [19] Niel de Beaudrap, Aleks Kissinger & Konstantinos Meichanetzidis (2020): *Satisfiability and Counting with Graphical Calculi*. *arXiv preprint arXiv:2004.06455*.
- [20] Ross Duncan & Kevin Dunne (2016): *Interacting Frobenius Algebras are Hopf*. In: *2016 31st Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, IEEE, pp. 1–10.
- [21] Brett Giles (2014): *An investigation of some theoretical aspects of reversible computing*. Ph.D. thesis, University of Calgary.
- [22] Michael Herrmann (2010): *Models of multipartite entanglement*. Ph.D. thesis, University of Oxford.
- [23] Chris Heunen (2013): *On the functor  $\ell^2$* . In: *Computation, Logic, Games, and Quantum Foundations. The Many Facets of Samson Abramsky*, Springer, pp. 107–121.
- [24] Kazuo Iwama, Yahiko Kambayashi & Shigeru Yamashita (2002): *Transformation rules for designing CNOT-based quantum circuits*. In: *Proceedings of the 39th annual Design Automation Conference*, pp. 419–424.
- [25] Stephen Lack (2004): *Composing props*. *Theory and Applications of Categories* 13(9), pp. 147–163.
- [26] Yves Lafont (2003): *Towards an algebraic theory of boolean circuits*. *Journal of Pure and Applied Algebra* 184(2-3), pp. 257–310.
- [27] Anthony Munson, Bob Coecke & Quanlong Wang (2019): *A note on AND-gates in ZX-calculus: QBC-completeness and phase gadgets*. *arXiv preprint arXiv:1910.06818*.
- [28] Robin Piedeleu (2018): *Picturing resources in concurrency*. Ph.D. thesis, University of Oxford.
- [29] Edmund Robinson & Giuseppe Rosolini (1988): *Categories of partial maps*. *Information and computation* 79(2), pp. 95–130.
- [30] Robert Rosebrugh & Richard J Wood (2002): *Distributive laws and factorization*. *Journal of Pure and Applied Algebra* 175(1-3), pp. 327–353.
- [31] Peter Selinger (2007): *Dagger compact closed categories and completely positive maps*. *Electronic Notes in Theoretical computer science* 170, pp. 139–163.
- [32] Quanlong Wang (2019): *ZX-calculus over arbitrary commutative rings and semirings*. *arXiv preprint arXiv:1912.01003*.
- [33] John van de Wetering & Sal Wolffs (2019): *Completeness of the Phase-free ZH-calculus*. *arXiv preprint arXiv:1904.07545*.

## A Proof of Lemma 2.14

Recall the statement of the Lemma:

**Lemma 2.14:** *The canonical functor  $\iota : \mathbb{X} \rightarrow \widetilde{\mathbb{X}}$  is faithful.*

*Proof.* Suppose that  $\iota(f) \sim \iota(g)$ , Then:



□

## B Proof of lemma 3.5

Recall the statement of the Lemma:

**Lemma 3.5** *Given a discrete inverse category  $\mathbb{X}$ ,  $c(\mathbb{X})$  and  $\widetilde{\mathbb{X}}$  are isomorphic as discrete Cartesian restriction categories.*

*Proof.* Define an identity on objects functor  $F : c(\mathbb{X}) \rightarrow \widetilde{\mathbb{X}}$  in the obvious way, sending the counits to the ancillary space. Similarly, define an identity on objects functor from  $G : \widetilde{\mathbb{X}} \rightarrow c(\mathbb{X})$  given by plugging counits into the ancillary space. These maps are clearly inverses to each other and preserve discrete Cartesian restriction structure; however, once again we must show that they are actually functors.

To see that  $F$  is a functor, it suffices to observe that every object in  $\widetilde{\mathbb{X}}$  is equipped with a counital Frobenius algebra, where the unit is in the image of the freely adjoined counit under  $F$ .

To prove that  $G$  is a functor, take some  $(f, S) \sim (g, T)$  in  $\tilde{\mathbb{X}}$ . Therefore, in  $\tilde{\mathbb{X}}$ , since the Frobenius structure is counital:

However, since the functor  $\mathbb{X} \rightarrow \tilde{\mathbb{X}}$  is faithful by Lemma 2.14, using the alternate equivalence relation of  $\tilde{\mathbb{X}}$  by Lemma 3.2, we have that in  $\mathbb{X}$ :

Therefore in  $c(\mathbb{X})$ :

So that combining the previous two equations:

□

## C Proof of Lemma 4.3

Recall the statement of the Lemma:

**Lemma 4.3:** Consider a finitely complete category  $\mathbb{X}$ , and a stable system of monics  $\mathcal{M}$ , so that  $\text{Span}^\sim(\mathcal{M})$  is a discrete inverse category with discrete Cartesian completion  $\text{Par}(\mathbb{X}, \mathcal{M})$ . Then adding a unit and counit to the chosen semi-Frobenius structure in  $\text{Span}^\sim(\mathcal{M})$  yields  $\text{Span}^\sim(\mathbb{X})$ .



*Proof.* First note that because  $\text{Par}(\mathbb{X}, \mathcal{M})$  is a discrete Cartesian restriction category, by Proposition 2.9,  $\text{Total}(\text{Par}(\mathbb{X}, \mathcal{M})) = \mathbb{X}$  is Cartesian. Therefore,  $\text{Span}^\sim(\mathbb{X})$  inherits a monoidal structure from the Cartesian product of  $\mathbb{X}$ : so that  $\text{Par}(\mathbb{X}, \mathcal{M})$  is a monoidal subcategory of  $\text{Span}^\sim(\mathbb{X})$ .

By assumption  $\text{Par}(\mathbb{X}, \mathcal{M})$  is the discrete Cartesian completion of  $\text{Span}^\sim(\mathbb{X})$ ; therefore by Lemma 3.5,  $\text{Par}(\mathbb{X}, \mathcal{M})$  can be presented by adding a counit to  $\text{Span}^\sim(\mathcal{M})$ . To compute the symmetric monoidal category with both a unit and counit adjoined is to compute a distributive law:

$$\text{Par}(\mathbb{X}, \mathcal{M}) \otimes_{\text{Span}^\sim(\mathcal{M})} \text{Par}^{\text{op}}(\mathbb{X}, \mathcal{M}) \rightarrow \text{Par}^{\text{op}}(\mathbb{X}, \mathcal{M}) \otimes_{\text{Span}^\sim(\mathcal{M})} \text{Par}(\mathbb{X}, \mathcal{M})$$

So that the unit and counit from  $\text{Par}^{\text{op}}(\mathbb{X}, \mathcal{M})$  and  $\text{Par}(\mathbb{X}, \mathcal{M})$ , are compatible with the symmetric monoidal structure of  $\text{Span}^\sim(\mathcal{M})$ . Note that this is implicitly asking for bimodule structures for both  $\text{Par}(\mathbb{X}, \mathcal{M})$  and  $\text{Par}^{\text{op}}(\mathbb{X}, \mathcal{M})$  over  $\text{Span}^\sim(\mathcal{M})$ . Since all 3 categories are subcategories of  $\text{Span}^\sim(\text{Set})$  with the same set of objects: the actions are given by pullback in  $\text{Span}^\sim(\mathbb{X})$ . So for example, tensoring  $\text{Par}(\mathbb{X}, \mathcal{M})$  by  $\text{Par}^{\text{op}}(\mathbb{X}, \mathcal{M})$  over  $\text{Span}^\sim(\mathcal{M})$  is given by the following coequalizer in  $\text{Span}(\text{Set})$ :

$$\text{Par}(\mathbb{X}, \mathcal{M}); \text{Span}^\sim(\mathcal{M}); \text{Par}^{\text{op}}(\mathbb{X}, \mathcal{M}) \rightrightarrows \text{Par}(\mathbb{X}, \mathcal{M}); \text{Par}^{\text{op}}(\mathbb{X}, \mathcal{M}) \rightarrow \text{Par}(\mathbb{X}, \mathcal{M}) \otimes_{\text{Span}^\sim(\mathcal{M})} \text{Par}^{\text{op}}(\mathbb{X}, \mathcal{M})$$

Given  $f, g$  maps in  $\text{Span}^\sim(\mathcal{M})$ , the distributive law is witnessed by the following equation:

Denote all actions and compositions by a semicolon, as they are all given by underlying pullback in  $\mathbb{X}$ . Given maps  $X \xleftarrow{f} A \xrightarrow{e} Y$  in  $\text{Par}^{\text{op}}(\mathbb{X}, \mathcal{M})$  and  $Y \xleftarrow{m} B \xrightarrow{g} Z$  in  $\text{Par}(\mathbb{X}, \mathcal{M})$ ; in the category given by quotienting by the distributive law, we have:

$$\begin{aligned} X \xleftarrow{f} A \xrightarrow{e} Y ; Y \xleftarrow{m} B \xrightarrow{g} Z &= X \xleftarrow{f} A = A ; A \xrightarrow{e} Y ; Y \xleftarrow{m} B = B ; A = B \xrightarrow{g} Z \\ &= X \xleftarrow{f} A = A ; \begin{array}{c} m' \swarrow C \searrow e' \\ A \xrightarrow{e} Y \xleftarrow{m} B \\ \sim \downarrow \\ A \xrightarrow{e} Y \xleftarrow{m} B \end{array} ; B = B \xrightarrow{g} Z \\ &= X \xleftarrow{f} A = A ; A \xleftarrow{m'} C \xrightarrow{e'} B ; B = B \xrightarrow{g} Z \\ &= X \xleftarrow{f} A = A ; A \xleftarrow{m'} C = C ; C \xrightarrow{e'} B ; B = B \xrightarrow{g} Z \\ &= X \xleftarrow{m'f} C = C ; C \xrightarrow{e'g} Z \end{aligned}$$

Therefore, every map in this quotiented category has a representative span in  $\mathbb{X}$ . This representative is unique; given two isomorphic maps  $X \xleftarrow{f'} B \xrightarrow{g'} Y \sim_h X \xleftarrow{f} A \xrightarrow{g} Y$  in  $\text{Span}^\sim(\mathbb{X})$ , we have:

$$\begin{aligned}
 X \xleftarrow{f'} B \xrightarrow{g'} Y &= X \xleftarrow{hf} B \xrightarrow{hg} Y \\
 &= X \xleftarrow{f} A \xrightarrow{h} B \xrightarrow{h} A \xrightarrow{g} Y \\
 &= X \xleftarrow{f} A \xrightarrow{h} B \xrightarrow{h} A \xrightarrow{g} Y \\
 &= X \xleftarrow{f} A \xrightarrow{h} B \xrightarrow{h} A \xrightarrow{g} Y \\
 &= X \xleftarrow{f} A \xrightarrow{h} B \xrightarrow{h} A \xrightarrow{g} Y \\
 &= X \xleftarrow{f} A \xrightarrow{h} B \xrightarrow{h} A \xrightarrow{g} Y \\
 &= X \xleftarrow{f} A \xrightarrow{h} B \xrightarrow{h} A \xrightarrow{g} Y \\
 &= X \xleftarrow{f} A \xrightarrow{h} B \xrightarrow{h} A \xrightarrow{g} Y
 \end{aligned}$$

So that, noting that  $\text{Span}^\sim(\mathbb{X})$ ,  $\text{Par}(\mathbb{X}, \mathcal{M})$ ,  $\text{Par}^{\text{op}}(\mathbb{X}, \mathcal{M})$  and  $\text{Span}^\sim(\mathcal{M})$  all have the same underlying symmetric monoidal structure, the category induced by this distributive law is isomorphic as a  $\dagger$ -symmetric monoidal category to  $\text{Span}^\sim(\mathbb{X})$ .  $\square$

## D Identities of TOF

Define the category TOF [8] to be the PROP, generated by the 1 ancillary bits  $|1\rangle$  and  $\langle 1|$  as well as the Toffoli gate, satisfying the identities given in Figure 5.

The Toffoli gate and the 1-ancillary bits allow cnot, not,  $|0\rangle$ ,  $\langle 0|$ , and flipped tof gate and flipped cnot gate can be defined in this setting:

$$\begin{aligned}
 \text{cnot} &:= \text{Toffoli with control on ancillary bit}, & \text{not} &:= \text{Toffoli with control on ancillary bit}, & \text{flipped tof} &:= \text{Toffoli with control on ancillary bit} \\
 \text{flipped cnot} &:= \text{Toffoli with control on ancillary bit}, & \text{ancillary bit} &:= \text{Toffoli with control on ancillary bit}, & \text{ancillary bit} &:= \text{Toffoli with control on ancillary bit}
 \end{aligned}$$

One can moreover construct generalized controlled not gates with arbitrarily many control wires in the obvious way. Let  $[x, X]$  denote a generalized Toffoli gate acting on the  $x$ th wire, controlled on the wires indexed by a set  $X$ . Then we can partially commute generalized controlled-not gates:

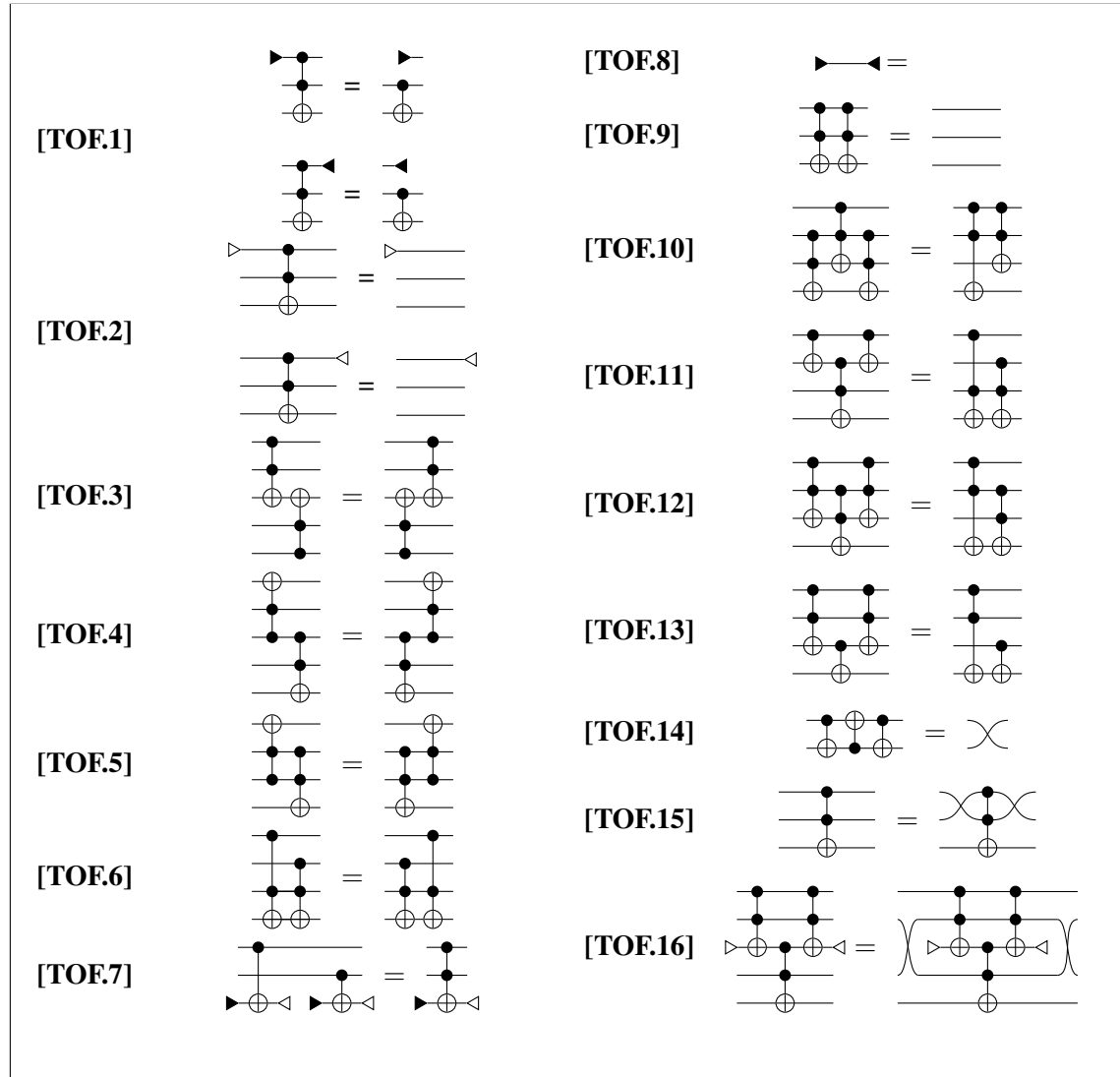


Figure 5: The identities of TOF

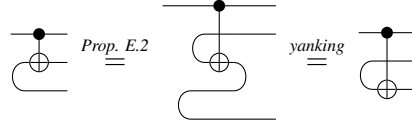
**Lemma D.1.** [18, Lem. 7.2.6] Let  $[x, X]$  and  $[y, Y]$  be generalized controlled not gates in TOF where  $x \notin Y$ . We can perform the identities of Iwama et al. [24], to commute them past each other with a trailing generalized controlled not gate as a side effect:

$$[y, X \cup Y][y, Y \sqcup \{x\}][x, X]$$



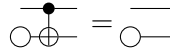
Therefore,

**Lemma E.3.**

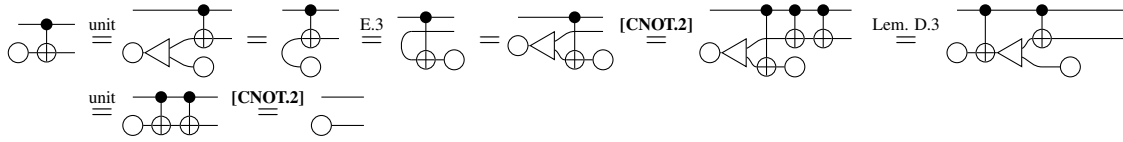


Thus

**Lemma E.4.**



*Proof.*



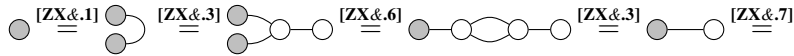
□

## E.2 Basic properties of ZX&

**Lemma E.5.**

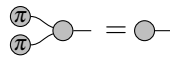


*Proof.*

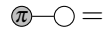


□

**Lemma E.6.** *The phase fusion of the black spider in ZX&,*



*in the presence of the other axioms is equivalent to asserting:*



*Or in other terms, the phase fusion of the black spider is equivalent to the interaction of the unit for and and the counit for copying as a bialgebra.*

*Proof.* For the one direction, suppose that phase fusion holds:

$$\pi \text{---} \bigcirc \stackrel{[\text{ZX}\&.3]}{=} \pi \text{---} \bigcirc \text{---} \bigcirc \stackrel{[\text{ZX}\&.1]}{=} \pi \text{---} \bigcirc \text{---} \bigcirc \text{---} \bigcirc \stackrel{[\text{ZX}\&.8]}{=} \begin{array}{c} \pi \\ \diagup \\ \bigcirc \text{---} \bigcirc \\ \diagdown \\ \pi \end{array} = \bigcirc \text{---} \bigcirc \stackrel{[\text{ZX}\&.7], \text{E.5}}{=}$$

Conversely if the unit part of the bialgebra rule holds:

$$\begin{array}{c} \pi \\ \diagup \\ \bigcirc \text{---} \bigcirc \\ \diagdown \\ \pi \end{array} \stackrel{[\text{ZX}\&.14]}{=} \pi \text{---} \bigcirc \text{---} \bigcirc \stackrel{[\text{ZX}\&.8]}{=} \pi \text{---} \bigcirc \text{---} \bigcirc = \bigcirc \text{---} \bigcirc$$

□

**Lemma E.7.**

$$\begin{array}{c} \bigcirc \\ \diagup \\ \& \\ \diagdown \\ \bigcirc \end{array} = \text{---} \bigcirc \text{---} \bigcirc$$

*Proof.*

$$\begin{array}{c} \bigcirc \\ \diagup \\ \& \\ \diagdown \\ \bigcirc \end{array} \stackrel{[\text{ZX}\&.1]}{=} \begin{array}{c} \pi \\ \diagup \\ \bigcirc \text{---} \bigcirc \\ \diagdown \\ \pi \end{array} \stackrel{[\text{ZX}\&.17]}{=} \begin{array}{c} \pi \\ \diagup \\ \& \\ \diagdown \\ \pi \end{array} \stackrel{[\text{ZX}\&.10]}{=} \text{---} \bigcirc \text{---} \bigcirc \stackrel{[\text{ZX}\&.8]}{=} \text{---} \bigcirc \text{---} \bigcirc$$

□

### E.3 Proof of Proposition 4.6

Recall the statement of Proposition 4.6:

**Proposition 4.6:** The interpretation  $\llbracket \_ \rrbracket_{\text{ZX}\&} : \text{ZX}\& \rightarrow \widehat{\text{TOF}}$  is a strict symmetric monoidal functor.

*Proof.* We prove that all of the axioms of ZX& hold in  $\widehat{\text{TOF}}$ :

**[ZX&.1]: Unitality:** By Lemma E.4:

$$\llbracket \begin{array}{c} \bigcirc \\ \diagup \\ \& \\ \diagdown \\ \bigcirc \end{array} \rrbracket_{\text{ZX}\&} = \begin{array}{c} \oplus \\ \diagup \\ \bigcirc \text{---} \bigcirc \\ \diagdown \\ \oplus \end{array} \stackrel{\text{comm.}}{=} \begin{array}{c} \oplus \\ \diagdown \\ \bigcirc \text{---} \bigcirc \\ \diagup \\ \oplus \end{array} \stackrel{\text{unit}}{=} \begin{array}{c} \bigcirc \text{---} \bigcirc \end{array} \stackrel{\text{Rem. E.1}}{=} \text{---} = \llbracket \text{---} \rrbracket_{\text{ZX}\&}$$

**Associativity:**

$$\llbracket \begin{array}{c} \bigcirc \\ \diagup \\ \& \\ \diagdown \\ \bigcirc \end{array} \rrbracket_{\text{ZX}\&} = \begin{array}{c} \oplus \\ \diagup \\ \bigcirc \text{---} \bigcirc \\ \diagdown \\ \oplus \end{array} \stackrel{[\text{CNOT.8}]}{=} \begin{array}{c} \oplus \\ \diagup \\ \bigcirc \text{---} \bigcirc \\ \diagdown \\ \oplus \end{array} \stackrel{\text{Rem. E.1}}{=} \begin{array}{c} \oplus \\ \diagup \\ \bigcirc \text{---} \bigcirc \\ \diagdown \\ \oplus \end{array}$$

$$= \left[ \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} \right]_{ZX\&}$$

**Frobenius:**

$$\left[ \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} \right]_{ZX\&} = \begin{array}{c} \oplus \\ \bullet \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} \xrightarrow{\text{Lem D.1}} \begin{array}{c} \oplus \\ \bullet \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} \xrightarrow{\text{Lem. E.4}} \begin{array}{c} \oplus \\ \bullet \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} \xrightarrow{[\text{CNOT.5}]} \begin{array}{c} \oplus \\ \bullet \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} = \left[ \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} \right]_{ZX\&}$$

**Phase amalgamation:**

$$\left[ \text{---} \pi \text{---} \pi \text{---} \right]_{ZX\&} = \text{---} \oplus \oplus \text{---} = \text{---} = \left[ \text{---} \right]_{ZX\&}$$

**[ZX&.2]:**

$$\left[ \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} \right]_{ZX\&} = \begin{array}{c} \oplus \\ \bullet \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} \xrightarrow{[\text{TOF.14}]} \begin{array}{c} \oplus \\ \bullet \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} \xrightarrow{[\text{CNOT.2}]} \begin{array}{c} \oplus \\ \bullet \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} \xrightarrow{\text{Lem. E.4}} \begin{array}{c} \oplus \\ \bullet \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} = \left[ \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} \right]_{ZX\&}$$

**[ZX&.3]:** This is immediate.

**[ZX&.4]:** This is immediate.

**[ZX&.5]:**

$$\left[ \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} \right]_{ZX\&} = \begin{array}{c} \oplus \\ \bullet \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} \xrightarrow{\text{Lem D.1}} \begin{array}{c} \oplus \\ \bullet \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} \xrightarrow{[\text{TOF.2}]} \begin{array}{c} \oplus \\ \bullet \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} \xrightarrow{\text{unit}} \begin{array}{c} \oplus \\ \bullet \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} \xrightarrow{[\text{CNOT.2}]} \begin{array}{c} \oplus \\ \bullet \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} \xrightarrow{\text{Lem. D.3}} \begin{array}{c} \oplus \\ \bullet \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} \xrightarrow{\text{Lem. E.4}} \begin{array}{c} \oplus \\ \bullet \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} \xrightarrow{[\text{TOF.14}]} \begin{array}{c} \oplus \\ \bullet \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} = \left[ \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} \right]_{ZX\&}$$

**[ZX&.6]:**

$$\left[ \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} \right]_{ZX\&} = \begin{array}{c} \oplus \\ \bullet \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} \xrightarrow{[\text{TOF.2}]} \begin{array}{c} \oplus \\ \bullet \end{array} \begin{array}{c} \oplus \\ \bullet \end{array} = \left[ \begin{array}{c} \text{---} \bullet \text{---} \\ \text{---} \bullet \text{---} \end{array} \right]_{ZX\&}$$

**[ZX&.7]:** This is immediate.

[ZX&amp;.8]:

$$\begin{aligned}
\llbracket \text{---} \circ \text{---} \circ \text{---} \rrbracket_{\text{ZX}\&} &= \text{---} \circ \text{---} \circ \text{---} \stackrel{\text{Lem. E.4}}{=} \text{---} \circ \text{---} \circ \text{---} \stackrel{[\text{TOF.14}]}{=} \text{---} \circ \text{---} \circ \text{---} \\
&= \text{---} \circ \text{---} \circ \text{---} = \llbracket \text{---} \circ \text{---} \rrbracket_{\text{ZX}\&}
\end{aligned}$$

[ZX&amp;.9]:

$$\begin{aligned}
\llbracket \text{---} \circ \text{---} \circ \text{---} \rrbracket_{\text{ZX}\&} &= \text{---} \circ \text{---} \circ \text{---} \stackrel{\text{Lem D.1}}{=} \text{---} \circ \text{---} \circ \text{---} \stackrel{\text{Rem. E.1}}{=} \text{---} \circ \text{---} \circ \text{---} \\
&\stackrel{[\text{TOF.2}]}{=} \text{---} \circ \text{---} \circ \text{---} \stackrel{[\text{TOF.2}]}{=} \text{---} \circ \text{---} \circ \text{---} \\
&\stackrel{\text{Lem D.1}}{=} \text{---} \circ \text{---} \circ \text{---} \stackrel{\text{Rem. E.1}}{=} \text{---} \circ \text{---} \circ \text{---} \\
&= \llbracket \text{---} \circ \text{---} \circ \text{---} \rrbracket_{\text{ZX}\&}
\end{aligned}$$

[ZX&amp;.10]:

$$\begin{aligned}
\llbracket \text{---} \circ \text{---} \circ \text{---} \rrbracket_{\text{ZX}\&} &= \text{---} \circ \text{---} \circ \text{---} \stackrel{\text{Lem. E.4}}{=} \text{---} \circ \text{---} \circ \text{---} \stackrel{[\text{TOF.1}]}{=} \text{---} \circ \text{---} \circ \text{---} \stackrel{\text{Rem. E.1}}{=} \text{---} \circ \text{---} \circ \text{---} \\
&= \llbracket \text{---} \circ \text{---} \rrbracket_{\text{ZX}\&}
\end{aligned}$$

[ZX&amp;.11]:

$$\llbracket \text{---} \circ \text{---} \circ \text{---} \rrbracket_{\text{ZX}\&} = \text{---} \circ \text{---} \circ \text{---} \stackrel{[\text{TOF.15}]}{=} \text{---} \circ \text{---} \circ \text{---} = \llbracket \text{---} \circ \text{---} \rrbracket_{\text{ZX}\&}$$

[ZX&amp;.12]:

$$\begin{aligned}
\llbracket \text{---} \circ \text{---} \circ \text{---} \rrbracket_{\text{ZX}\&} &= \text{---} \circ \text{---} \circ \text{---} \stackrel{[\text{TOF.4}]}{=} \text{---} \circ \text{---} \circ \text{---} \stackrel{\text{unit}}{=} \text{---} \circ \text{---} \circ \text{---} \\
&\stackrel{[\text{TOF.2}]}{=} \text{---} \circ \text{---} \circ \text{---} \stackrel{\text{Lem D.1}}{=} \text{---} \circ \text{---} \circ \text{---} = \llbracket \text{---} \circ \text{---} \rrbracket_{\text{ZX}\&}
\end{aligned}$$



$$\left[ \left[ \text{AND} \right] \right]_{\text{ZX}\&} = \text{[TOF,2]} = \text{Rem. E.1} = \left[ \left[ \text{XOR} \right] \right]_{\text{ZX}\&}$$
$$\left[ \left[ \pi \right] \text{---} \bigcirc \right]_{\text{ZX}\&} = \text{---} \begin{array}{c} \bullet \\ \text{---} \\ \bigoplus \end{array} \text{---} \stackrel{[\text{TOF.1}]}{=} \text{---} \begin{array}{c} \text{---} \\ \text{---} \end{array} = \left[ \begin{array}{c} \pi \\ \pi \end{array} \right]_{\text{ZX}\&}$$
[illegible]

The figure illustrates a sequence of quantum circuit transformations for a 4-qubit system. The initial circuit (top left) features a multi-controlled NOT gate (labeled  $ZX\&$ ) and several CNOT gates. The sequence of transformations is as follows:

- Step 1:** Initial circuit with a multi-controlled NOT gate and CNOTs.
- Step 2:** Transformation labeled  $\text{Lem. D.1}$  and  $\text{Rem. E.1}$ .
- Step 3:** Transformation labeled  $\text{Rem. E.1}$  and  $[\text{TOF.2}]$ .
- Step 4:** Transformation labeled  $[\text{TOF.9}]$  and  $\text{Rem. E.1}$ .
- Step 5:** Transformation labeled  $[\text{TOF.2}]$  and  $[\text{TOF.2}]$ .
- Step 6:** Transformation labeled  $[\text{ZX}\&.11]$ .
- Step 7:** Transformation labeled  $\text{Lem. D.1}$ .
- Step 8:** Final simplified circuit, which is the dual of the initial one, labeled  $ZX\&$ .

☐

### E.4 Proof of Proposition 4.7

Recall the statement of Proposition 4.7:

**Proposition 4.7:** *The interpretation  $\llbracket \cdot \rrbracket_{\widehat{\text{TOF}}} : \widehat{\text{TOF}} \rightarrow \text{ZX}\&$  is a strict symmetric monoidal functor.*

*Proof.* First, observe:

$$\begin{aligned} \llbracket \text{---} \oplus \text{---} \rrbracket_{\widehat{\text{TOF}}} &= \text{Diagram 1} \stackrel{\text{[ZX}\&.14]}{=} \text{Diagram 2} \stackrel{\text{[ZX}\&.1]}{=} \text{Diagram 3} \\ &\stackrel{\text{Lem. E.5, [ZX}\&.7]}{=} \text{Diagram 4} \stackrel{\text{[ZX}\&.10]}{=} \text{Diagram 5} \stackrel{\text{[ZX}\&.4]}{=} \text{Diagram 6} \end{aligned}$$

Thus:

$$\llbracket \text{---} \oplus \text{---} \rrbracket_{\widehat{\text{TOF}}} = \text{Diagram 1} = \text{Diagram 2} \stackrel{\text{[ZX}\&.14]}{=} \text{Diagram 3} \stackrel{\text{[ZX}\&.1]}{=} \text{Diagram 4} \stackrel{\text{Lem. E.5, [ZX}\&.7]}{=} \text{Diagram 5}$$

Thus:

$$\llbracket \text{---} \oplus \text{---} \rrbracket_{\widehat{\text{TOF}}} = \text{Diagram 1} = \text{Diagram 2} \stackrel{\text{[ZX}\&.1]}{=} \text{Diagram 3}$$

We prove that all of the axioms of  $\widehat{\text{TOF}}$  hold in  $\text{ZX}\&$  :

[TOF.1]:

$$\begin{aligned} \llbracket \text{---} \oplus \text{---} \rrbracket_{\widehat{\text{TOF}}} &= \text{Diagram 1} \stackrel{\text{[ZX}\&.14]}{=} \text{Diagram 2} \stackrel{\text{[ZX}\&.10]}{=} \text{Diagram 3} \\ &\stackrel{\text{[ZX}\&.3]}{=} \text{Diagram 4} = \llbracket \text{---} \oplus \text{---} \rrbracket_{\widehat{\text{TOF}}} \end{aligned}$$

[TOF.2]:

$$\begin{aligned} \llbracket \text{---} \oplus \text{---} \rrbracket_{\widehat{\text{TOF}}} &= \text{Diagram 1} \stackrel{\text{[ZX}\&.6]}{=} \text{Diagram 2} \stackrel{\text{Lem. E.7}}{=} \text{Diagram 3} \\ &\stackrel{\text{[ZX}\&.1]}{=} \text{Diagram 4} \stackrel{\text{[ZX}\&.3]}{=} \text{Diagram 5} = \llbracket \text{---} \oplus \text{---} \rrbracket_{\widehat{\text{TOF}}} \end{aligned}$$

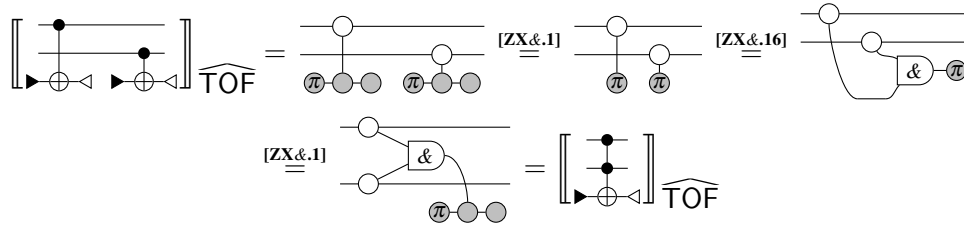
[TOF.3]: This follows from the spider law.

[TOF.4]: This follows from the spider law.

[TOF.5]: This follows from the spider law.

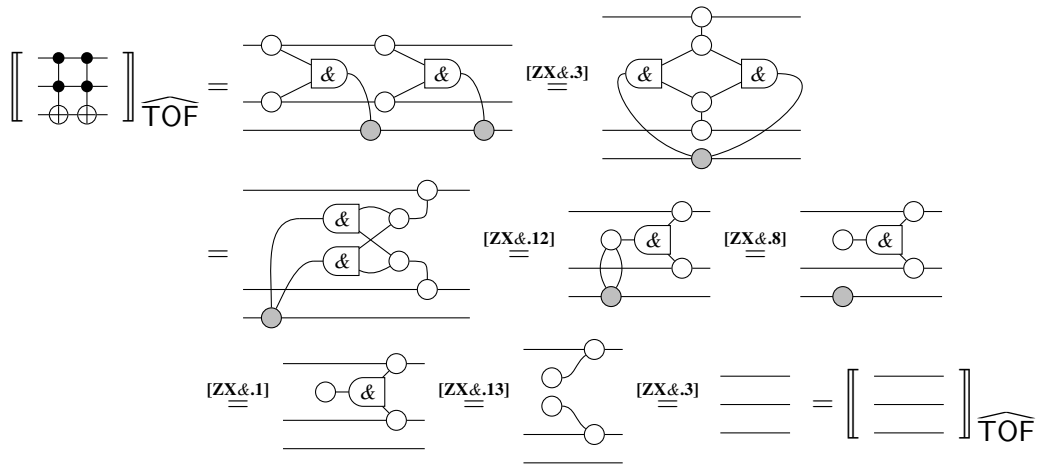
[TOF.6]: This follows from the spider law.

[TOF.7]:

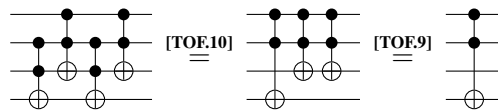


[TOF.8]: This follows immediately from Lemma E.5 and [ZX&.7].

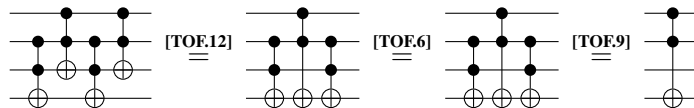
[TOF.9]:



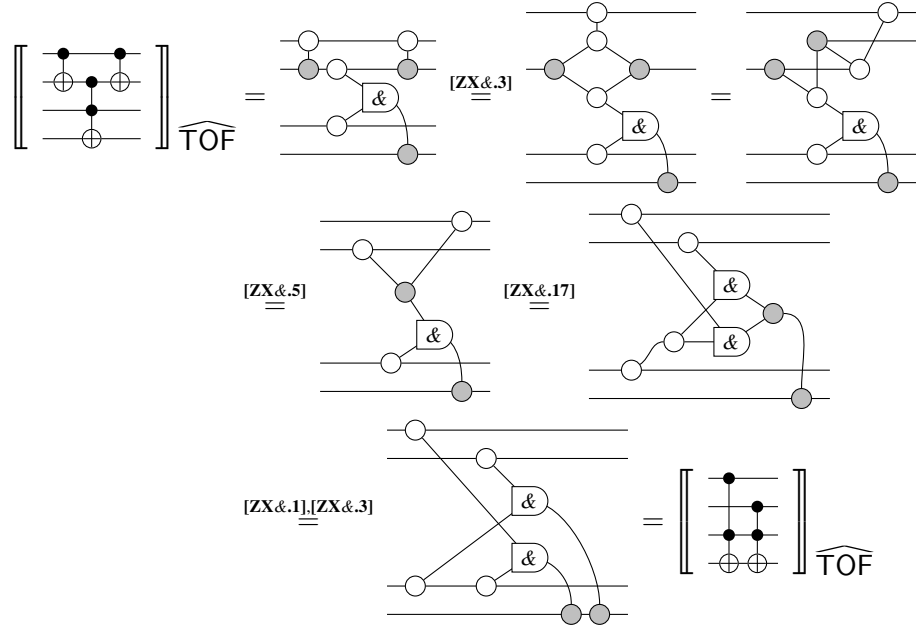
[TOF.10]: It is easier to prove that [TOF.10] is redundant. Given [TOF.9], [TOF.6] and [TOF.12], [TOF.10] is equivalent to the following:



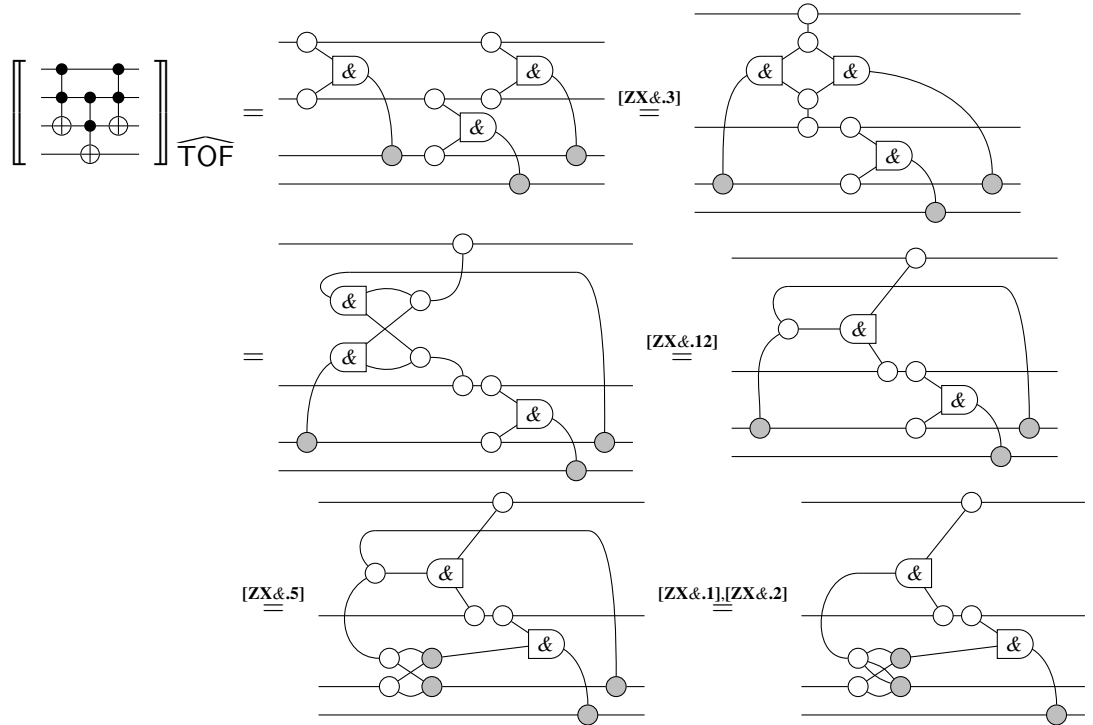
However

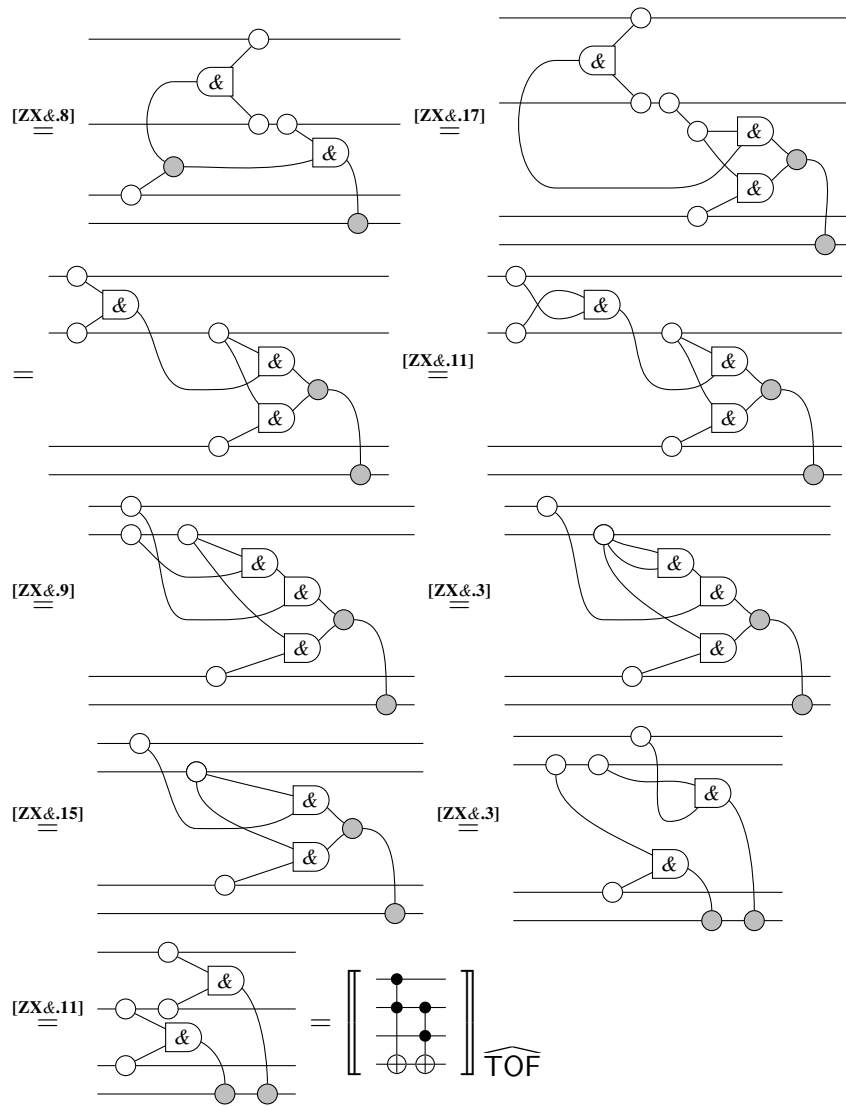


[TOF.11]:

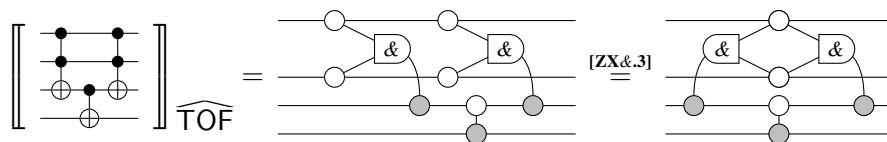


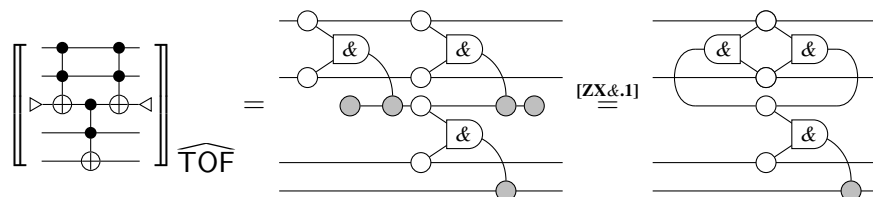
[TOF.12]:

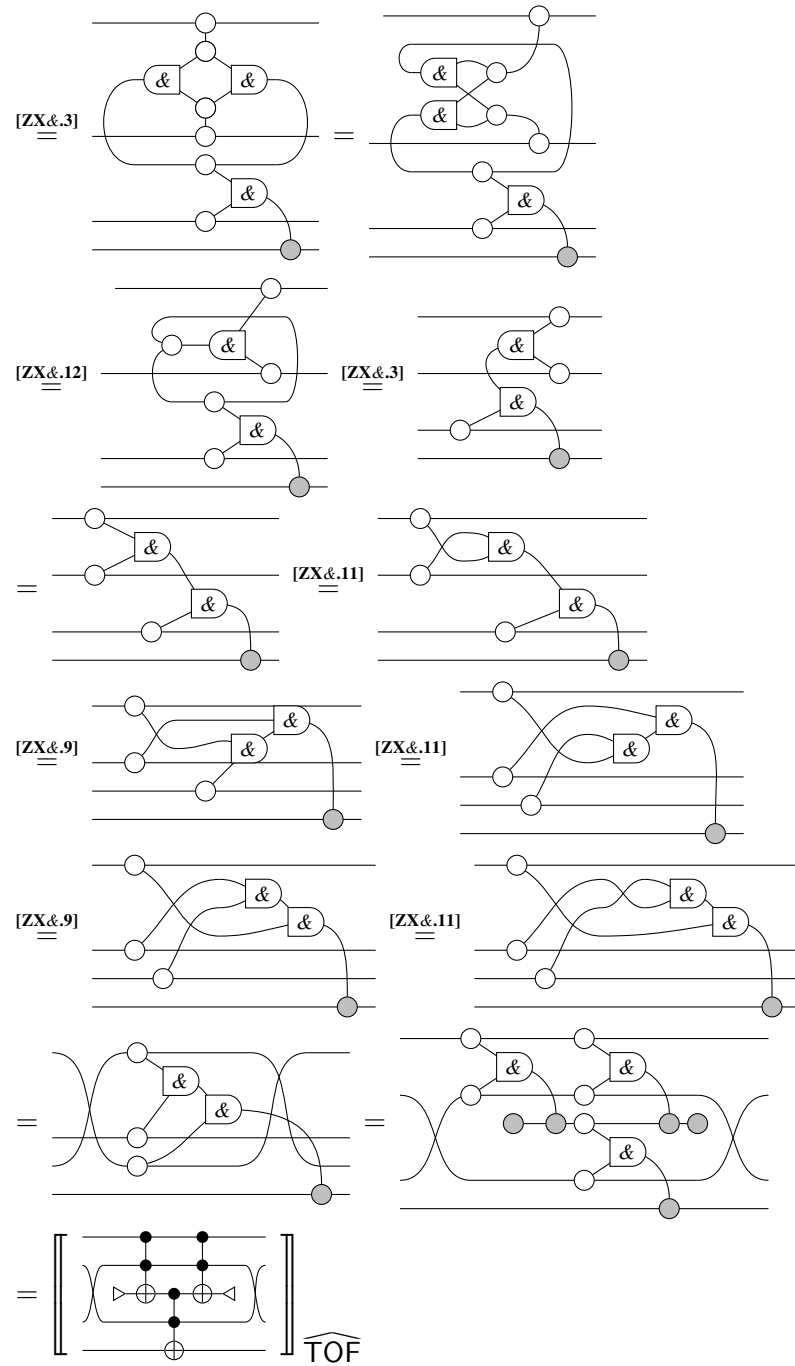




[TOF.13]:







Where unitality and counitality follow from the fact that the white spiders are Frobenius algebras. Also, we must also note that both Frobenius algebras induce the same compact closed structure, as is implied by the spider law; this is immediate.

□

### E.5 Proof of Theorem 4.8

**Theorem 4.8** *The interpretations  $\llbracket - \rrbracket_{\text{ZX}\&}$  and  $\llbracket - \rrbracket_{\widehat{\text{TOF}}}$  are inverses, so that  $\widehat{\text{TOF}}$  and ZX& are isomorphic as strongly compact closed props.*

*Proof.* First we show that  $\llbracket \llbracket - \rrbracket_{\text{ZX}\&} \rrbracket_{\widehat{\text{TOF}}} = 1$ :

**For the white spider:** The case for the unit and counit is trivial. For the (com)multiplication we have:

$$\llbracket \llbracket \text{white spider} \rrbracket_{\text{ZX}\&} \rrbracket_{\widehat{\text{TOF}}} = \llbracket \text{white spider} \rrbracket_{\widehat{\text{TOF}}} = \text{white spider with } \pi \text{ phases} = \text{white spider} = \text{white spider}$$

**For the grey spider:** The cases for the unit, counit and  $\pi$  phase are trivial. For the (com)multiplication we have:

$$\llbracket \llbracket \text{grey spider} \rrbracket_{\text{ZX}\&} \rrbracket_{\widehat{\text{TOF}}} = \llbracket \text{grey spider} \rrbracket_{\widehat{\text{TOF}}} = \text{grey spider with } \pi \text{ phases} = \text{grey spider} = \text{grey spider}$$

**For the and gate:**

$$\llbracket \llbracket \text{and gate} \rrbracket_{\text{ZX}\&} \rrbracket_{\widehat{\text{TOF}}} = \llbracket \text{and gate} \rrbracket_{\widehat{\text{TOF}}} = \text{and gate with } \pi \text{ phases} = \text{and gate} = \text{and gate}$$

Next, we show that  $\llbracket \llbracket - \rrbracket_{\widehat{\text{TOF}}} \rrbracket_{\text{ZX}\&} = 1$ : The ancillae are trivial. For the Toffoli gate:

$$\begin{aligned} \llbracket \llbracket \text{Toffoli gate} \rrbracket_{\widehat{\text{TOF}}} \rrbracket_{\text{ZX}\&} &= \llbracket \text{Toffoli gate} \rrbracket_{\text{ZX}\&} \\ &\stackrel{\text{Lem. D.1}}{=} \text{Toffoli gate with ancillae} \stackrel{[\text{TOF.2}]}{=} \text{Toffoli gate with ancillae} \stackrel{\text{unit}}{=} \text{Toffoli gate with ancillae} \\ &\stackrel{\text{Lem. D.1}}{=} \text{Toffoli gate with ancillae} \stackrel{[\text{TOF.2}]}{=} \text{Toffoli gate with ancillae} \stackrel{\text{unit}}{=} \text{Toffoli gate with ancillae} \end{aligned}$$

□



## F The identities of CNOT

The category CNOT [9] is the  $\dagger$ -symmetric monoidal subcategory of TOF generated by the controlled not gate and ancillary bits  $|1\rangle, \langle 1|$ . A complete set of identities is presented in the following figure, because some of the identities are used in the translation between ZX& and the (co)unital completion of TOF.

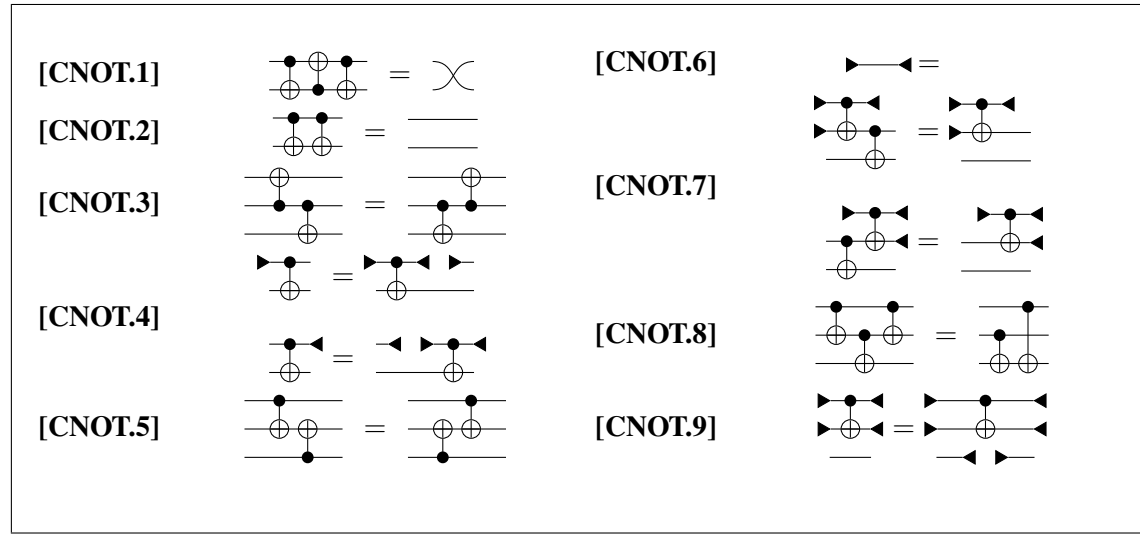


Figure 6: The identities of CNOT